

Finite atomic lattices and their monomial ideals *

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Abstract

This paper primarily studies monomial ideals by their associated lcm-lattices. It first introduces notions of weak coordinatizations of finite atomic lattices which have weaker hypotheses than coordinatizations and shows the characterizations of all such weak coordinatizations. It then defines a finite super-atomic lattice in $\mathcal{L}(n)$, investigates the structures of $\mathcal{L}(n)$ by their super-atomic lattices and proposes an algorithm to calculate all the super-atomic lattices in $\mathcal{L}(n)$. It finally presents a specific labeling of finite atomic lattice and obtains the conditions that the specific labelings of finite atomic lattices are the weak coordinatizations or the coordinatizations by using the terminology of super-atomic lattices.

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1 Introduction

Let M be a monomial ideal in a polynomial ring $R = K[x_1, x_2, \dots, x_n]$ where K is a field. We are interested in studying a minimal free resolution of R/M , and specifically understanding the maps in this resolution (see [1, 4, 6, 13, 14]). For a monomial ideal M , the minimal resolution is completely dependent on the information in the lcm-lattice of M , or $\text{LCM}(M)$, which is the lattice of least common multiples of the minimal generators of M partially ordered by divisibility. In 1999, Gasharov, Peeva, and Welker in [7] expressed the multigraded Betti numbers of R/M using the homology groups of certain open intervals in $\text{LCM}(M)$. They further showed that the combinatorial type of minimal resolutions of

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a monomial ideal is determined by its *LCM* lattice. In 2006, Phan in [12] proved that all finite atomic lattices can be realized as the *LCM* lattice of some monomial ideal M . He gave a construction which is motivated by the observation that for any coordinatization of an atomic lattice as a monomial ideal the set of lattice elements for which a given variable has a given degree bound is an order ideal. Essentially, he identified which order ideals are necessary and labels them with variables. In 2009, Mapes gave a generalization of the main construction in [12] to describe all monomial ideals with a given *LCM* lattice, i.e., she proved a statement as below (see [9], also [10]).

Any labeling \mathcal{M} of elements in a finite atomic lattice P by monomials satisfying the following two conditions will yield a coordinatization of the lattice P .

(A1) If $p \in \text{mi}(P)$ then $m_p \neq 1$ (i.e., all meet-irreducibles are labeled).

(A2) If $\gcd(m_p, m_q) \neq 1$ for some $p, q \in P$ then p and q must be comparable (i.e., each variable only appears in monomials along one chain in P).

Mapes thought that it would be interesting to give an explicit formulation for when two coordinatizations are equivalent in this sense or to prove a version of the result above which has weaker hypotheses. This question has been inadvertently answered by Lukas Katthän in [8] using different terminology. Additionally, Mapes and Piechnik recently posted a paper on the arXiv [11] which contains a result which is equivalent to Katthän's (see Proposition 3.2 of [11]). However, all of them do not give a general construction of the labeling \mathcal{M} which does not satisfy the conditions (A1) and (A2) but \mathcal{M} is a coordinatization.

On the other hand, the fact that the set of finite atomic lattices on n ordered atoms, denoted by $\mathcal{L}(n)$, is itself a finite atomic lattice leads us to the question: what is the relationship between minimal resolutions of coordinatizations of lattices in $\mathcal{L}(n)$? The answer, due to a result in [7], is that the total Betti numbers are weakly monotonic along chains in $\mathcal{L}(n)$. This inspires us to understand the structure of $\mathcal{L}(n)$. In 2013, Mapes in [10] proved that for any relation $P > Q$ in $\mathcal{L}(n)$ there exists a coordinatization of Q producing a monomial ideal M_Q and a deformation of exponents of M_Q such that the lcm-lattice of the deformed ideal is P .

This paper will continue the topics on describing all monomial ideals by their *LCM* lattices and understanding the structure of $\mathcal{L}(n)$, which is organized as follows. In Section 2, we give some preliminaries for convenience. In Section 3, we introduce notions of weak coordinatizations of finite atomic lattices and show their characterizations. In Section 4, we define a finite super-atomic lattice in $\mathcal{L}(n)$, investigate the structures of $\mathcal{L}(n)$ by their super-atomic lattices and propose an algorithm to calculate all the super-atomic lattices in $\mathcal{L}(n)$. In the end, we present a specific labeling of finite atomic lattice and obtain the conditions which are used to determine whether the specific labelings are the weak coordinatizations or the coordinatizations by terminology of super-atomic lattices.

2 Preliminaries

Definition 2.1 ([5]) A partially ordered set is a system consisting of a nonempty set P and a binary relation \leq in P such that the following conditions are satisfied for all $x, y, z \in P$:

- (i) $x \leq x$.
- (ii) If $x \leq y$ and $y \leq x$, then $x = y$.
- (iii) If $x \leq y$ and $y \leq z$, then $x \leq z$.

The relation \leq is a partial order in the set P , and P is said to be partially ordered by the relation \leq . Again, the partially ordered set P is denoted as (P, \leq) .

If x and y are elements of the partially ordered set P , we also write $y \geq x$ in case $x \leq y$. We write $x < y$ if $x \leq y$ and $x \neq y$, and we say that x is less than y . Again, when $x < y$, we write $y > x$ and say that y is greater than x . The formulas $x \not\leq y$ and $y \not\leq x$ both mean that $x \leq y$ does not hold, again we write $x \parallel y$ if $x \not\leq y$ and $y \not\leq x$, and we say that x and y are not comparable. In addition, if $x < y$ and there is no element $z \in P$ such that $x < z < y$, then we say that x is covered by y (or y covers x), and we write $x \prec y$ (or $y \succ x$).

Definition 2.2 ([10]) A lattice is a set $(P, <)$ with an order relation $<$ which is transitive and antisymmetric satisfying the following properties:

- (1) P has a maximum element denoted by 1.
- (2) P has a minimum element denoted by 0.
- (3) Every pair of elements a and b in P has a join $a \vee b$ which is the least upper bound of the two elements.
- (4) Every pair of elements a and b in P has a meet $a \wedge b$ which is the greatest lower bound of the two elements.

If P only satisfies conditions (2) and (4) then it is a meet-semilattice, and if P only satisfies conditions (1) and (3) then it is a join-semilattice. Furthermore, if P is a meet-semilattice with a unique maximal element then it is a lattice. Equivalently, if P is a join-semilattice with a unique minimal element then it is a lattice.

We define an atom of a lattice P to be an element $x \in P$ such that x covers 0. We denote the set of atoms in P by $\text{atoms}(P)$ (see [5, 10]). Let A and B be two sets. We define $A - B = \{x \in A : x \notin B\}$.

Definition 2.3 ([10]) If P is a lattice and every element in $P - \{0\}$ is the join of atoms, then P is an atomic lattice. Furthermore, if P is finite, then it is a finite atomic lattice.

If P is a lattice, then we define an element $x \in P$ to be meet-irreducible if $x \neq a \wedge b$ for any $a > x, b > x$. We denote the set of meet-irreducible elements in P by $\text{mi}(P)$. Given an element $x \in P$, an order ideal of x is defined to be the set $[x] = \{a \in P : a \leq x\}$. Similarly, we define a filter of x to be $[x] = \{a \in P : x \leq a\}$ (see [5, 10]).

Lemma 2.1 (**Lemma 2.3 of [10]**) *Let P be a finite atomic lattice. Every element $p \in P$ is the meet of all the meet-irreducible elements l such that $l \geq p$.*

It will be convenient to consider finite atomic lattices as sets of sets in the following way. Let \mathcal{S} be a set of subsets of $\{1, \dots, n\}$ with no duplicates, closed under intersections, and containing the entire set, the empty set, and the sets $\{i\}$ for all $1 \leq i \leq n$. Then it is easy to see that \mathcal{S} is a finite atomic lattice by ordering the sets in \mathcal{S} by inclusion.

Conversely, it is clear that any finite atomic lattice P can be expressed in this way, simply by letting

$$\mathcal{S}_P = \{\sigma : \sigma = \text{supp}(p), p \in P\},$$

where $\text{supp}(p) = \{a_i : a_i \leq p, a_i \in \text{atoms}(P)\}$ (see [2, 3, 10]).

Definition 2.4 ([7]) The *LCM* lattice, $LCM(M)$, of a monomial ideal M is the set of least common multiples of minimal generators of M , partially ordered by divisibility.

Customarily, we denote that $\text{lcm}\emptyset = 1$ and $\text{gcd}\emptyset = 1$.

Example 2.1 For the monomial ideal $M = (a^2cd, abd, abc) \subseteq k[a, b, c, d]$ the Hasse diagram of the *LCM* lattice of M is shown as Fig.1 (note the minimal element of the lattice has been left off, as will often be the case).

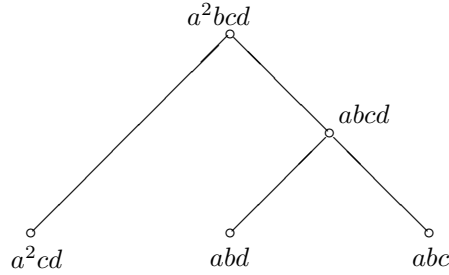


Fig.1. The lattice $LCM(M)$

One conclusion in [7] is that for monomial ideals all minimal resolutions are completely dependent on the information in the *LCM* lattice. Specifically, one can compute multigraded Betti numbers using the *LCM* lattice $LCM(M)$ and all ideals with a given *LCM* lattice have isomorphic minimal free resolutions.

Definition 2.5 ([9]) Define a labeling of a finite atomic lattice P to be any assignment of non-trivial monomials $\mathcal{M} = \{m_{p_1}, \dots, m_{p_t}\}$ to some set of elements $p_i \in P$. It will be convenient to think of unlabeled elements as having the label 1. Define a monomial ideal $M_{P,\mathcal{M}}$ to be the ideal generated by monomials

$$x(a) = \prod_{p \in [a]^c} m_p \quad (1)$$

for each $a \in \text{atoms}(P)$ where $[a]^c$ means taking the complement of $[a]$ in P . We say that the labeling \mathcal{M} is a coordinatization if the lcm-lattice of $M_{P,\mathcal{M}}$ is isomorphic to P .

Lemma 2.2 (Proposition 3.2.1 of [9] and Theorem 3.2 of [10],) Any labeling \mathcal{M} of elements in a finite atomic lattice P by monomials satisfying the following two conditions will yield a coordinatization of the lattice P .

(A1) If $p \in \text{mi}(P)$ then $m_p \neq 1$ (i.e., all meet-irreducibles are labeled).

(A2) If $\text{gcd}(m_p, m_q) \neq 1$ for some $p, q \in P$ then p and q must be comparable (i.e., each variable only appears in monomials along one chain in P).

Note that if the labeling \mathcal{M} satisfies the conditions (A1) and (A2) then the isomorphic map f from P to $LCM(M_{P,\mathcal{M}})$ must be that

$$f(p) = \prod_{q \in [p]^c} m_q \quad (2)$$

for any $p \in P$.

Lemma 2.3 (Lemma 3.3 of [10]) *If $p \in [q]^c$ for some $p, q \in P$ where P is a finite atomic lattice, then $[p] \subseteq [q]^c$.*

Let M be a monomial ideal with n generators and let P_M be its lcm-lattice. For notational purposes, denote P_M as the set consisting of elements denoted \bar{p} which represent the monomials occurring in P_M . Now, define an abstract finite atomic lattice P where the elements in P are formal symbols p satisfying the relations $p < p'$ if and only if $\bar{p} < \bar{p}'$ in P_M . In other words, P is the finite atomic lattice isomorphic to P_M obtained by simply forgetting the data of the monomials in P_M . Define a labeling of P in the following way, let \mathcal{D} be the set consisting of monomials m_p for each $p \in P$ defined by

$$m_p = \frac{\gcd\{\bar{t} : t > p\}}{\bar{p}}, \quad (3)$$

where by convention $\gcd\{\bar{t} : t > p\}$ for $p = 1$ is defined to be $\bar{1}$. Note that m_p is a monomial since clearly \bar{p} divides \bar{t} for all $t > p$.

Lemma 2.4 (Proposition 3.6 of [10]) *Given M a monomial ideal with lcm-lattice P_M . If P is an abstract finite atomic lattice where P is isomorphic to P_M as lattices then the labeling \mathcal{D} of P as defined by (3) is a coordinatization and the resulting monomial ideal $M_{P,\mathcal{D}} = M$.*

Although Lemma 2.4 shows that the labeling \mathcal{D} of P as defined by (3) is a coordinatization, the following theorem will further verify that the labeling \mathcal{D} induced by (3) is the same as \mathcal{M} if \mathcal{M} satisfies the conditions of Lemma 2.2.

Theorem 2.1 *Let \mathcal{M} be a labeling of a finite atomic lattice P which satisfies the conditions of Lemma 2.2. Then the labeling \mathcal{D} of P as defined by (3) satisfies $m'_p = m_p$ for each $p \in P$ where $m_p \in \mathcal{M}$ and $m'_p \in \mathcal{D}$.*

Proof. Suppose that P has n atoms. Then by Lemma 2.2 and equation (2), we have that $\bar{p} = f(p) = \prod_{q \in [p]^c} m_q$ for any $p \in P$. Thus the formula (3) implies that

$$\begin{aligned} m'_p &= \frac{\gcd\{\prod_{q \in [t]^c} m_q : t > p\}}{\prod_{q \in [p]^c} m_q} \\ &= \frac{\prod_{q \in [p]^c} m_q * \gcd\{\prod_{q \in [t]^c - [p]^c} m_q : t > p\}}{\prod_{q \in [p]^c} m_q} \\ &= \gcd\left\{ \prod_{q \in [t]^c - [p]^c} m_q : t > p \right\}. \end{aligned}$$

On the other hand, if $a, b \in P$ and $a \geq b$, then $[a]^c \supseteq [b]^c$. Thus $\prod_{q \in [b]^c - [p]^c} m_q \mid \prod_{q \in [a]^c - [p]^c} m_q$, which implies that

$$m'_p = \gcd\left\{\prod_{q \in [t]^c - [p]^c} m_q : t \succ p\right\} = \gcd\left\{\prod_{q \in [t]^c - [p]^c} m_q : t \succ p\right\}.$$

This follows that $m'_p = m_p * \gcd\{\prod_{q \in [t]^c - [p]^c - \{p\}} m_q : t \succ p\}$ since $p \in [t]^c - [p]^c$ for any $t \succ p$.

Thus in order to prove $m_p = m'_p$ for any $p \in P$, we just need to show that

$$\gcd\left\{\prod_{q \in [t]^c - [p]^c - \{p\}} m_q : t \succ p\right\} = 1$$

as follows.

(a) If there is only one element $t \in P$ satisfying $t \succ p$, then $[t]^c - [p]^c - \{p\} = \emptyset$. Otherwise, there exists an element $d \in P$ such that $d \succ p$ and $d \not\geq t$, where $d \not\geq t$ implies that $d < t$ or $d \parallel t$. If $d < t$ then $t \succ d = p$ since $t \succ p$, a contradiction. If $d \parallel t$ then there exists an element $c \leq d$ such that $c \succ p$ since $d \succ p$. Thus $c = t$ since there is only one element covering p , a contradiction. Therefore, we have that $\gcd\{\prod_{q \in [t]^c - [p]^c - \{p\}} m_q : t \succ p\} = \gcd \emptyset = 1$.

(b) Suppose that there are k elements t_1, t_2, \dots, t_k in P such that $t_i \succ p$ for any $1 \leq i \leq k$ where $k \geq 2$. If $\gcd\{\prod_{q \in [t]^c - [p]^c - \{p\}} m_q : t \succ p\} \neq 1$, then there exists a variable x_p such that $x_p \mid \gcd\{\prod_{q \in [t]^c - [p]^c - \{p\}} m_q : t \succ p\}$. Therefore, we have an element $q_i \succ p$ and $q_i \not\geq t_i$ such that $x_p \mid m_{q_i}$ for any $1 \leq i \leq k$. By the second condition of Lemma 2.2, we have that q_1, q_2, \dots, q_k are comparable, i.e. $\{q_1, q_2, \dots, q_k\}$ lies in a chain in P . Hence, there exists an element $1 \leq r \leq k$ such that $\{q_1, q_2, \dots, q_k, t_r\}$ be a chain in P , and then for any $1 \leq j \leq k$ we have $q_j \geq t_r$ since $q_j \succ p$ and $t_r \succ p$. Thus $q_r \geq t_r$, a contradiction. Therefore, $\gcd\{\prod_{q \in [t]^c - [p]^c - \{p\}} m_q : t \succ p\} = 1$. \square

3 Weak coordinatizations

One of the main results in [12] is that every finite atomic lattice is in fact the lcm-lattice of a monomial ideal. In 2009, Mapes in [9] introduced a definition of coordinatization. Moreover, she proved that there are some specific constructions which produce a monomial ideal whose lcm-lattice has a given lattice structure, i.e., Lemma 2.2 (see also [10]). Mapes thought that it would be interesting to give an explicit formulation for when two coordinatizations are equivalent in this sense or to prove a version of Lemma 2.2 which has weaker hypotheses.

In this section, we shall introduce the notion of a weak coordinatization which has weaker hypotheses than Definition 2.5, and show a sufficient condition which yields the weak coordinatization.

Definition 3.1 Let \mathcal{M} be a coordinatization of a finite atomic lattice P . If for each $a \in \text{atoms}(P)$ the map $g : P \rightarrow LCM(M_{P, \mathcal{M}})$ satisfying $g(a) = x(a)$ is isomorphic, then we say \mathcal{M} is a strong coordinatization.

It is easy to see that if the labeling \mathcal{M} satisfies the conditions (A1) and (A2) then it is a strong coordinatization by the formulas (1) and (2). Note that a coordinatization may not be strong. For example, let P be the finite atomic lattice with a labeling \mathcal{M} as Fig.2.

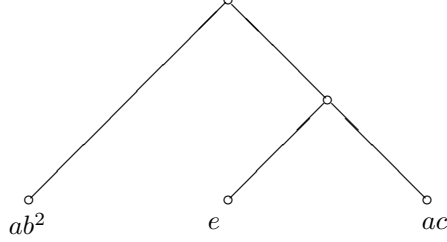


Fig.2. A finite lattice P with a labeling \mathcal{M}

Using (1), we know that $M_{P,\mathcal{M}} = (ace, a^2b^2c, ab^2e)$. It is clear that $LCM(M_{P,\mathcal{M}}) \cong P$. Then \mathcal{M} is a coordinatization of P . However, it is not a strong coordinatization by Definition 3.1.

Let P be a finite atomic lattice and $p \in P$. Define $B_p = \{T \subseteq \text{supp}(p) : \bigvee_{b \in T} b = p\}$.

Definition 3.2 Let \mathcal{M} be a labeling of a finite atomic lattice P . Define a monomial ideal $I_{P,\mathcal{M}}$ to be the ideal generated by monomials

$$\Delta(a) = \gcd\{\text{lcm}\{x(b) : b \in T\} : T \in \bigcup_{p \geq a} B_p\} \quad (4)$$

for each $a \in \text{atoms}(P)$. We say that the labeling \mathcal{M} is a weak coordinatization if for each $a \in \text{atoms}(P)$ the map $g : P \rightarrow LCM(I_{P,\mathcal{M}})$ satisfying $g(a) = \Delta(a)$ is isomorphic.

We first have the following lemma.

Lemma 3.1 A labeling \mathcal{M} is a strong coordinatization of a finite atomic lattice P if and only if it is a weak coordinatization and $\Delta(a) = x(a)$ for any $a \in \text{atoms}(P)$.

Proof. By Definitions 3.1 and 3.2, the sufficiency is clear, and we just need to prove the necessity.

First, by equation (4) it is easy to see that $\Delta(a) \mid x(a)$ since $\{a\} \in \bigcup_{p \geq a} B_p$. Secondly,

let g be the isomorphic map from P to $LCM(M_{P,\mathcal{M}})$. Then

$$g(p) = \text{lcm}\{x(b) : b \in \text{supp}(p)\} = \text{lcm}\{x(b) : b \in T\}$$

for any $p \in P$ and any $T \in B_p$. If $a \leq p$ then $g(a) = x(a) \leq g(p)$ (i.e. $g(a) = x(a) \mid g(p)$) since g is isomorphic. Then $x(a) \mid \text{lcm}\{x(b) : b \in T\}$ for all $T \in \bigcup_{p \geq a} B_p$. Thus we

have $x(a) \mid \Delta(a)$ by (4). Therefore, we have $\Delta(a) = x(a)$, i.e., $I_{P,\mathcal{M}} = M_{P,\mathcal{M}}$. Thus $LCM(I_{P,\mathcal{M}}) = LCM(M_{P,\mathcal{M}})$. Finally, \mathcal{M} is also a weak coordinatization of P since \mathcal{M} is a strong coordinatization by Definition 3.2. \square

However, a weak coordinatization of a finite atomic lattice P needs not to be a coordinatization of P . For instance, let P be the finite atomic lattice with a labeling as Fig.3. Then by Definitions 2.5 and 3.2,

$$M_{P,\mathcal{M}} = (b^2c^2d^2e^2, acd^2e^2, a^2b^2d^2e^2, a^2b^3c^2e, a^2b^3c^2d),$$

$$I_{P,\mathcal{M}} = (b^2c^2d^2e^2, acd^2e^2, a^2b^2d^2e^2, a^2b^2c^2e, a^2b^2c^2d).$$

Then it is obvious that the lattice $LCM(I_{P,\mathcal{M}})$ shown as Fig.4 is isomorphic to P , which means that the labeling \mathcal{M} is a weak coordinatization of P by Definition 3.2. On the other hand, the lattice $LCM(M_{P,\mathcal{M}})$ shown as Fig.5 is not isomorphic to P , i.e., the labeling \mathcal{M} is not a coordinatization of P by Definition 2.5.

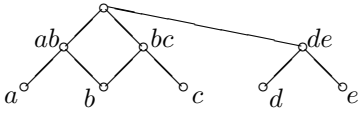


Fig.3. P with a labeling

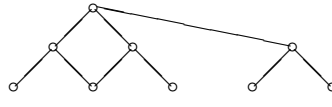


Fig.4. $LCM(I_{P,\mathcal{M}})$

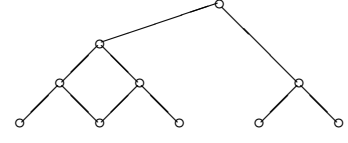


Fig.5. $LCM(M_{P,\mathcal{M}})$

Lemma 3.2 *Let \mathcal{M} be a labeling of a finite atomic lattice P . If $p \in P$ and an atom b is in $\text{supp}(p) - R$ where $R \in B_p$, then $\Delta(b) \mid \text{lcm}\{\Delta(r) : r \in R\}$.*

Proof. Suppose $\Delta(b) \nmid \text{lcm}\{\Delta(r) : r \in R\}$. Then there exists a monomial x^{u_b} such that $x^{u_b} \nmid \text{lcm}\{\Delta(r) : r \in R\}$ where x^{u_b} is the highest power of x dividing $\Delta(b)$. Let $S = \{a \in R : x^{u_b} \mid x(a)\}$. Assume that x^{u_a} is the highest power of x dividing $\Delta(a)$ for any $a \in S$. Thus we have $u_a < u_b$ since $x^{u_b} \nmid \text{lcm}\{\Delta(r) : r \in R\}$. Again, it follows from (4) that for any $a \in S$ there exists an element $q_a \in P$ with $q_a \geq a$ and a set $T_a \in B_{(q_a)}$ such that x^{u_a} is the highest power of x dividing $\text{lcm}\{x(t) : t \in T_a\}$. Thus

$$x^{u_b} \nmid \text{lcm}\{x(t) : t \in T_a\} \quad (5)$$

for each $a \in S$.

Next, we let $C = \bigcup_{a \in S} T_a \cup (R - S)$. Clearly, we have

$$\bigvee_{c \in C} c = \bigvee_{a \in S} (\bigvee T_a) \vee \bigvee (R - S) = \bigvee_{a \in S} q_a \vee \bigvee (R - S) \geq \bigvee_{a \in S} a \vee \bigvee (R - S) = p \geq b$$

and $C \in B_{(\bigvee_{c \in C} c)}$. Using (4), we have $\Delta(b) \mid \text{lcm}\{x(c) : c \in C\}$. Thus

$$x^{u_b} \mid \text{lcm}\{x(c) : c \in C\}. \quad (6)$$

However, from (5) we know that if $c \in \bigcup_{a \in S} T_a$ then $x^{u_b} \nmid x(c)$ since $u_a < u_b$ and x^{u_a} is the highest power of x dividing $\text{lcm}\{x(t) : t \in T_a\}$. Moreover, if $c \in R - S$ then $x^{u_b} \nmid x(c)$ by the construction of set S . Hence, $x^{u_b} \nmid \text{lcm}\{x(c) : c \in C\}$, contrary to (6). Therefore, $\Delta(b) \mid \text{lcm}\{\Delta(r) : r \in R\}$. \square

Lemma 3.3 *Let \mathcal{M} be a labeling of a finite atomic lattice P . If there exists a variable x_0 such that $x_0 \mid m_p$ and $x_0 \mid m_q$ imply that p and q are comparable, then $x_0 \nmid \frac{x(a)}{\text{gcd}(\Delta(a), x(a))}$ for any atom a in P .*

Proof. Let $S = \{s \in P : x_0 \nmid m_s\}$ and $R = P - S$. Suppose that $\overline{m_s} = x_s$ for any $s \in S$ and $\overline{m_r} = x_0^r$ where x_0^r is the highest power of x_0 dividing m_r for any $r \in R$. Clearly, the labeling $\overline{M} = \{\overline{m_p} : p \in P\}$ satisfies the conditions of Lemma 2.2. That is to see \overline{M} is a strong coordinatization of P . Hence, by Lemma 3.1, we know that \overline{M} is a weak coordinatization of P and

$$\overline{x(a)} = \overline{\Delta(a)} \quad (7)$$

for any atom $a \in \text{atoms}(P)$ where $\overline{x(a)} \in M_{P, \overline{M}}$ and $\overline{\Delta(a)} \in I_{P, \overline{M}}$.

Now, assume that $x_0^{a_1}$ and $x_0^{\overline{a_1}}$ are highest powers of x_0 dividing $\overline{x(a)}$ and $\overline{x(a)}$, respectively, and $x_0^{a_2}$ and $x_0^{\overline{a_2}}$ are highest powers of x_0 dividing $\Delta(a)$ and $\overline{\Delta(a)}$, respectively. By Definition 2.5, we have $a_1 = \overline{a_1}$ which together with equation (4) implies that $\overline{a_2} = a_2$.

Using (7), we have $\overline{a_1} = \overline{a_2}$. Therefore, $a_2 = a_1$, i.e., $x_0 \nmid \frac{x(a)}{\gcd(\Delta(a), x(a))}$. \square

Theorem 3.1 *Any labeling \mathcal{M} of elements in a finite atomic lattice P by monomials satisfying the following two conditions will yield a weak coordinatization of the lattice P .*

(C1) *If $p \in \text{mi}(P)$ then $m_p \neq 1$.*

(C2) *If $\gcd(m_p, m_q) \neq 1$ for some $p, q \in P$ then either p and q must be comparable, or*

$$r(p) = \frac{m_p}{\gcd(m_p, m_q)} \neq 1 \text{ and } r(q) = \frac{m_q}{\gcd(m_p, m_q)} \neq 1$$

and if x, y are in $\{s \in P - \{q\} : \gcd(m_p, m_s) \neq 1\}$ or $\{s \in P - \{p\} : \gcd(m_q, m_s) \neq 1\}$, then x and y are comparable.

Proof. The proof of Theorem 3.1 is made in several steps. Let P' be the lcm-lattice of $I_{P, \mathcal{M}}$. Then the main part is to show that P' is isomorphic to P . For $b \in P$ define $g : P \rightarrow P'$ to be the map such that

$$g(b) = \text{lcm}\{\Delta(a_i) : a_i \in \text{supp}(b)\}.$$

Clearly, the map g is well-defined. Next, we shall show that g is an isomorphism of lattices.

A. The map g is a bijection on atoms.

It is enough that the set of minimal generators for $I_{P, \mathcal{M}}$ has the same cardinality as $\text{atoms}(P)$. In other words we must show $\Delta(a) \neq \Delta(b)$ for $a, b \in \text{atoms}(P)$ with $a \neq b$.

In fact, from Lemma 2.1 we know that $\text{mi}(P) \cap [a] \neq \text{mi}(P) \cap [b]$, $\text{mi}(P) \cap [a] \not\subseteq \text{mi}(P) \cap [b]$ and $\text{mi}(P) \cap [b] \not\subseteq \text{mi}(P) \cap [a]$. Thus sets $\text{mi}(P) \cap [a]^c$ and $\text{mi}(P) \cap [b]^c$ are distinct and one can not be a subset of the other. Hence there exists at least an element $q \in \text{mi}(P) \cap [a]^c$ but $q \notin \text{mi}(P) \cap [b]^c$.

Since q is meet-irreducible, condition (C1) implies that $m_q \neq 1$. Let x_q be a variable satisfying $x_q \mid m_q$. Suppose $p \in [b]^c$ satisfying m_p is divided by x_q . Again, $p \neq q$ since $q \notin \text{mi}(P) \cap [b]^c$. The proof is split into two parts as follows:

(I) If p and q are comparable, then either $q < p$ or $p < q$. If $q < p$ then we have $q \in [p]$, and $q \in [p] \subseteq [b]^c$ by $p \in [b]^c$ and Lemma 2.3, a contradiction. So that $p < q$

for any $p \in [b]^c$ with $x_q \mid m_p$. Let C_q be a chain in P consisting of elements whose monomial labels are divisible by x_q . Assume that $z \in C_q \cap [b]^c$, then $x_q \mid m_z$, and $z < q \in [a]^c$, i.e., $z \in [a]^c$ by Lemma 2.3. Thus $C_q \cap [b]^c \subseteq [a]^c$. Further, $C_q \cap [b]^c \subsetneq [a]^c$ since $q \in [a]^c$ but $q \notin C_q \cap [b]^c$. Hence by the equation (1), if x_q^s is the highest power of x_q dividing $x(a)$ then $x_q^s \nmid x(b)$. Again, by Lemma 3.3 we have that x_q^s is the highest power of x_q dividing $\Delta(a)$. Therefore, $\Delta(a) \nmid \Delta(b)$ since $\Delta(b) \mid x(b)$ and $x_q^s \nmid x(b)$.

(II) Suppose that $p \parallel q$. Then for any $x, y \in \{s \in P - \{p\} : \gcd(m_q, m_s) \neq 1\}$, we have that x and y are comparable by condition (C2). Thus t and q are comparable if $\gcd(m_q, m_t) \neq 1$ with $t \in [b]^c - \{p\}$.

Next, let $C_q^* = \{u \in P : x_q \mid m_u\}$ where $x_q \mid r(q)$. We claim that $p \notin C_q^*$. Indeed, if $p \in C_q^*$, then $x_q \mid r(q)$ and $x_q \mid m_p$, i.e., $x_q \mid \gcd(r(q), m_p)$. Again, we have $\gcd(r(q), m_p) = 1$ by condition (C2). Thus $x_q \mid 1$, a contradiction. Hence z and q are comparable for any $z \in C_q^* \cap [b]^c$ since t and q are comparable if $\gcd(m_t, m_q) \neq 1$ with $t \in [b]^c - \{p\}$.

If $z = q$ then $q \in [b]^c$, contrary to $q \notin [b]^c$. If $z > q$ then $q \in [b]^c$ by Lemma 2.3, a contradiction. Thus $z < q$, which implies that $z \in [a]^c$ by using $q \in [a]^c$ and Lemma 2.3. Therefore, $C_q^* \cap [b]^c \subseteq C_q^* \cap [a]^c$. Note that $q \in C_q^* \cap [a]^c$ and $q \notin C_q^* \cap [b]^c$. Hence

$$C_q^* \cap [b]^c \subsetneq C_q^* \cap [a]^c. \quad (8)$$

We claim that C_q^* is a chain, i.e., u and v are comparable for any $u, v \in C_q^*$. By $p \notin C_q^*$ and the construction of the set C_q^* , we have $C_q^* \subseteq \{s \in P - \{p\} : \gcd(m_q, m_s) \neq 1\}$ which implies that u and v are comparable for any $u, v \in C_q^*$. Thus the variable x_q satisfies the conditions of Lemma 3.3. Therefore, $x_q \nmid \frac{x(a)}{\gcd(\Delta(a), x(a))}$.

Below, assume that x_q^s is the highest power of x_q dividing $x(a)$. Then we know that x_q^s is also the highest power of x_q dividing $\Delta(a)$. Again, $x_q^s \nmid x(b)$ by (1) and (8). Therefore, $\Delta(a) \nmid \Delta(b)$ since $\Delta(b) \mid x(b)$ and $x_q^s \mid \Delta(a)$.

Finally, from (I) and (II) we have that $\Delta(a) \neq \Delta(b)$ for $a, b \in \text{atoms}(P)$ with $a \neq b$, i.e., the map g is a bijection on atoms.

B. The map g is meet-preserving.

Clearly, $\text{supp}(p \wedge q) = \text{supp}(p) \cap \text{supp}(q)$ for any pair $p, q \in P$. Then $g(p \wedge q) = \text{lcm}\{\Delta(a_u) : a_u \in \text{supp}(p) \cap \text{supp}(q)\} = g(p) \wedge g(q)$ for any pair $p, q \in P$, i.e., g is meet-preserving.

C. The map g is join-preserving.

If p and q are two elements of P , then $\text{supp}(p) \cup \text{supp}(q) \subseteq \text{supp}(p \vee q)$ since $p \leq p \vee q$ and $q \leq p \vee q$. Let $T_{(p \vee q)} = \text{supp}(p \vee q) - \text{supp}(p) \cup \text{supp}(q)$. Clearly, if $T_{(p \vee q)} = \emptyset$ then $g(p \vee q) = g(p) \vee g(q)$ since $g(p \vee q) = g(p) \vee g(q) \vee \text{lcm}\{\Delta(a_v) : a_v \in T_{(p \vee q)}\}$ and $\text{lcm}\emptyset = 1$. Now, suppose that $T_{(p \vee q)} \neq \emptyset$. By Lemma 3.2, we have

$$\text{lcm}\{\Delta(a_v) : a_v \in T_{(p \vee q)}\} \leq \text{lcm}\{\Delta(a_v) : a_v \in \text{supp}(p) \cup \text{supp}(q)\}.$$

Therefore, $g(p \vee q) = g(p) \vee g(q)$ i.e., the map g is join-preserving.

D. The map g is surjective.

For any $p' \in P'$, we have that $p' = \text{lcm}\{\Delta(a_i) : i \in I\}$ and $|I| < \infty$. Let $b = \bigvee_{i \in I} a_i \in P$. Then we have

$$g(b) = \text{lcm}\{\Delta(a_i) : a_i \in \text{supp}(b)\} = \text{lcm}\{\Delta(a_i) : i \in I\} = p'$$

by Lemma 3.2. Therefore, the map g is surjective.

E. The map g is injective.

Equivalently, we need to prove that $g(a) = g(b)$ if and only if $a = b$ for any $a, b \in P$. Clearly, if $0 \in \{a, b\}$ and $g(a) = g(b)$ then $g(a) = g(b) = g(0) = 1$, which implies that $a = b = 0$. Next, we suppose that $a, b \in P - \{0\}$. The proof will be completed by two parts.

(i) a and b are comparable. In this case, we first suppose that $a < b$ and $g(a) = g(b)$. Then by the definition of map g , we have that

$$g(b) = \text{lcm}\{\Delta(a_i) : a_i \in \text{supp}(a)\} \vee \text{lcm}\{\Delta(a_j) : a_j \in \text{supp}(b) - \text{supp}(a)\},$$

i.e.,

$$g(b) = g(a) \vee \text{lcm}\{\Delta(a_j) : a_j \in \text{supp}(b) - \text{supp}(a)\}. \quad (9)$$

On the other hand, from $a < b$ we have that $\text{supp}(b) - \text{supp}(a) \neq \emptyset$ and $[a]^c \subsetneq [b]^c$. Thus there exists an atom a_j such that $a_j \leq b$ but $a_j \not\leq a$. Then $a_j \parallel a$, which means that $[a_j]^c \cap \text{mi}(P) \neq [a]^c \cap \text{mi}(P)$, i.e., there exists an element q satisfying $q \in [a_j]^c \cap \text{mi}(P)$ but $q \notin [a]^c \cap \text{mi}(P)$. Clearly, $[a_i]^c \cap \text{mi}(P) \subseteq [a]^c \cap \text{mi}(P)$ for any $a_i \in \text{supp}(a)$. Thus, $q \notin [a_i]^c \cap \text{mi}(P)$ for any $a_i \in \text{supp}(a)$. It is easy to see that q is non-trivially labeled by condition (C1). Therefore, by the proof of A there exists a variable x_q with the highest power as x_q^t such that $x_q^t \mid \Delta(a_j)$, and $x_q^t \nmid \Delta(a_i)$ for any $a_i \in \text{supp}(a)$. Thus $x_q^t \mid \text{lcm}\{\Delta(a_j) : a_j \in \text{supp}(b) - \text{supp}(a)\}$ and $x_q^t \nmid g(a)$. Therefore, by (9) we have that $x_q^t \mid g(b)$, then $g(b) \nmid g(a)$, contrary to $g(a) = g(b)$.

Similarly, we can rule out that $b < a$.

Consequently, $a = b$ if $g(a) = g(b)$ in the case that a and b are comparable.

(ii) a and b are not comparable. We easily see that

$$g(b) = \text{lcm}\{\Delta(a_i) : a_i \in \text{supp}(b) \cap \text{supp}(a)\} \vee \text{lcm}\{\Delta(a_i) : a_i \in \text{supp}(b) - \text{supp}(a)\}. \quad (10)$$

On the other hand, $\text{supp}(b) - \text{supp}(a) \neq \emptyset$ since $a \parallel b$. Clearly, for any $a_i \in \text{supp}(b) - \text{supp}(a)$ we have that $a_i \parallel a$. Thus there exists an element $q \in [a_i]^c \cap \text{mi}(P)$ such that $q \notin [a]^c \cap \text{mi}(P)$. Further, if $a_j \in \text{supp}(a)$ then $[a_j]^c \cap \text{mi}(P) \subseteq [a]^c \cap \text{mi}(P)$ since $a_j \leq a$. Hence $q \notin [a_j]^c \cap \text{mi}(P)$ for any $a_j \in \text{supp}(a)$. With analogous proof to (i), using (10) we can prove that $g(b) \neq g(a)$, a contradiction.

Finally, with (i) and (ii) we know that the map g is injective. \square

The following example will illustrate Theorem 3.1.

Example 3.1 Let P be a finite atomic lattice with a labeling as Fig.6. It is easy to see that the labeling of P satisfies the conditions of Theorem 3.1 and does not satisfy the conditions of Lemma 2.2.

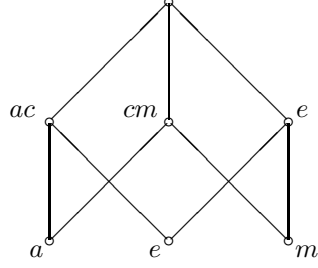


Fig.6. The lattice P with labeling \mathcal{M}

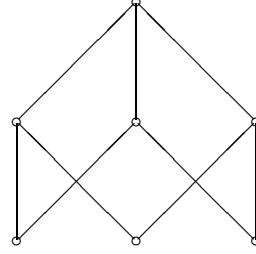


Fig.7. $LCM(I_{P,\mathcal{M}})$

One can clarify that $I_{P,\mathcal{M}} = (e^2m, acm^2, a^2ce)$, and $LCM(I_{P,\mathcal{M}})$ is isomorphic to P (see Figs.6 and 7).

Remark 3.1 From Theorem 2.1, if the monomial ideal $M = M_{P,\mathcal{M}}$ with the labeling \mathcal{M} satisfying the conditions of Lemma 2.2, then $\mathcal{D} = \mathcal{M}$. On the other hand, by Lemma 2.4 we know that if the monomial ideal $M = I_{P,\mathcal{M}}$ with the labeling \mathcal{M} satisfies the conditions of Theorem 3.1 and does not satisfy the conditions of Lemma 2.2, then M must induce a new labeling \mathcal{D} which is different from \mathcal{M} and the resulting monomial ideal $M_{P,\mathcal{D}} = I_{P,\mathcal{M}} = M$.

4 Finite super-atomic lattices

Let $\mathcal{L}(n)$ be the set of all finite atomic lattices with n ordered atoms. This set $\mathcal{L}(n)$ has a partial order where $Q \leq P$ if and only if there exists a join-preserving map which is a bijection on atoms from P to Q (note that such a map will also be surjective)(see [10]). In this section, we shall discuss the structure of lattice $\mathcal{L}(n)$. We shall first define a finite super-atomic lattice, and then find out all the finite super-atomic lattices in $\mathcal{L}(n)$.

Definition 4.1 A finite atomic lattice P is called super-atomic if it satisfies that for any $p \in P - \text{atoms}(P) \cup \{0\}$ and any $T \in B_p$, there exist exactly two elements $a, b \in T$ such that $a \vee b = p$.

For example, the finite atomic lattice P shown as Fig.8 is super-atomic.

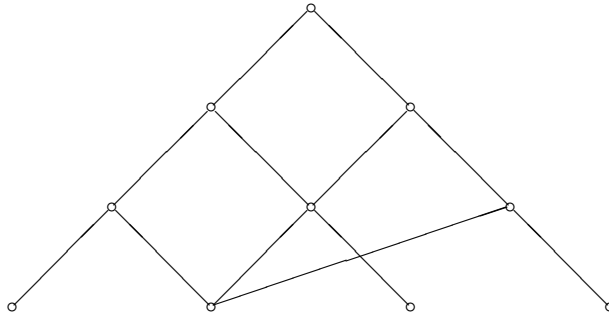


Fig.8. A finite super-atomic lattice

Theorem 4.1 *Let P be a finite atomic lattice. Then P is super-atomic if and only if for any $p \in P - \text{atoms}(P) \cup \{0\}$, there exist two atoms p_1, p_2 in P such that $p = p_1 \vee p_2$ satisfying that $\text{supp}(p) - \{p_1\} \in \mathcal{S}_P$ and $\text{supp}(p) - \{p_2\} \in \mathcal{S}_P$.*

Proof. Suppose that there exist two atoms p_1, p_2 in P such that $p = p_1 \vee p_2$ for certain $p \in P - \text{atoms}(P) \cup \{0\}$ with $\text{supp}(p) - \{p_2\} \notin \mathcal{S}_P$. We first note that $(\mathcal{S}_P, \subseteq)$ is the lattice corresponding to P . Then

$$\text{supp}(p) \supseteq \bigvee_{r \in \text{supp}(p) - \{p_2\}} \{r\} \supsetneq \text{supp}(p) - \{p_2\}.$$

Thus $\bigvee_{r \in \text{supp}(p) - \{p_2\}} \{r\} = \text{supp}(p)$, i.e., $\bigvee_{r \in \text{supp}(p) - \{p_2\}} r = p$, which together with Definition 4.1 implies that there exist two atoms $t, s \in \text{supp}(p) - \{p_2\}$ such that $t \vee s = p$. Therefore, $p_1 \vee p_2 = t \vee s = p$ and $\{p_1, p_2\} \neq \{s, t\}$, contrary to Definition 4.1. So $\text{supp}(p) - \{p_2\} \in \mathcal{S}_P$. Similarly, we have that $\text{supp}(p) - \{p_1\} \in \mathcal{S}_P$.

Below we shall show that P is super-atomic. By the hypothesis, for $p \in P - \text{atoms}(P) \cup \{0\}$ there exist two atoms p_1, p_2 in P such that $p = p_1 \vee p_2$ satisfying $\text{supp}(p) - \{p_1\} \in \mathcal{S}_P$ and $\text{supp}(p) - \{p_2\} \in \mathcal{S}_P$. Thus for any $T \in B_p$, if $\{p_1, p_2\} \not\subseteq T$ then either $\bigvee_{q \in T} \{q\} \subseteq \text{supp}(p) - \{p_2\}$ or $\bigvee_{q \in T} \{q\} \subseteq \text{supp}(p) - \{p_1\}$. In any case we have $\bigvee_{q \in T} q < p$, contrary to $\bigvee_{q \in T} q = p$. Therefore,

$$\{p_1, p_2\} \subseteq T \text{ for any } T \in B_p. \quad (11)$$

Suppose there exist two atoms q_1, q_2 with $\{q_1, q_2\} \neq \{p_1, p_2\}$ such that $q_1 \vee q_2 = p$, then $\{p_1, p_2\} \not\subseteq \{q_1, q_2\}$, contrary to (11) since $\{q_1, q_2\} \in B_p$. Consequently, for any $T \in B_p$ there exist exactly two elements $p_1, p_2 \in T$ such that $p_1 \vee p_2 = p$, i.e., P is a finite super-atomic lattice by Definition 4.1. \square

Next, it will be convenient to think of the elements in $\mathcal{L}(n)$ as set \mathcal{S} as described in Section 2. By Theorem 4.1, we have a lemma as below.

Lemma 4.1 *If \mathcal{S} is a finite super-atomic lattice in $\mathcal{L}(n)$ with $n \geq 2$. Then the following statements hold:*

- (A1) $\mathcal{T} = \{\emptyset, \{1\}, \{2\}, \dots, \{n\}, \{1, 2, \dots, n\}\} \subseteq \mathcal{S}$.
- (A2) If $S \in \mathcal{S} - \mathcal{S}_0 \cup \mathcal{S}_1$ where $\mathcal{S}_0 = \{\emptyset\}$ and $\mathcal{S}_1 = \{\{1\}, \{2\}, \dots, \{n\}\}$, then there exist two elements $\{i\}, \{j\} \in \mathcal{S}_1$ such that $S = \{i\} \vee \{j\}$. Moreover, $S - \{r\} \in \mathcal{S}$ if and only if $r \in \{i, j\}$.
- (A3) Let $S_1 = \{u\} \vee \{v\}$ and $S_2 = \{k\} \vee \{h\}$ in \mathcal{S} . If $S_1 \parallel S_2$ then $\{u, v\} \not\subseteq S_2$ and $\{k, h\} \not\subseteq S_1$.

In what follows, we shall suggest an algorithm to construct a finite super-atomic lattice in $\mathcal{L}(n)$ with $n \geq 2$. Let S be a set with $|S| \geq 2$, and $\delta(S)$ be a subset of S which exactly has two elements, say $\delta(S) = \{i_S, j_S\}$.

Algorithm 4.1

Input: $X = \{1, 2, \dots, n\}$.

Output: \mathcal{S}^* .

Step 1. Take $\mathcal{S}_0 = \{\emptyset\}, \mathcal{S}_1 = \{\{1\}, \{2\}, \dots, \{n\}\}, \mathcal{S}_n = \{X\}, \mathcal{S}^* = \mathcal{S}_0 \cup \mathcal{S}_1 \cup \mathcal{S}_n$ and $k := 0$.

Step 2. If $n - k = 2$, then go to Step 7.

Step 3. For any $S \in \mathcal{S}_{n-k}$, take $\delta(S) = \{i_S, j_S\}$ satisfying $\delta(S) \not\subseteq T$ for any $T \in \mathcal{S}_{n-k} - \{S\}$.

Step 4. Take $\mathcal{S}_{n-k-1} = \bigcup_{S \in \mathcal{S}_{n-k}} \{S - \{i_S\}, S - \{j_S\}\}$.

Step 5. $k := k + 1$.

Step 6. $\mathcal{S}^* := \mathcal{S}^* \cup \mathcal{S}_{n-k}$, and go to Step 2.

Step 7. Stop.

Theorem 4.2 Every output $(\mathcal{S}^*, \subseteq)$ in Algorithm 4.1 is a finite super-atomic lattice in $\mathcal{L}(n)$. Further, every finite super-atomic lattice in $\mathcal{L}(n)$ can be constructed by Algorithm 4.1.

Proof. Throughout the proof, we denote $\bigvee \delta(S) = \{i_S\} \vee \{j_S\}$ for any $S \in \mathcal{S}^*$. Next, we shall prove that every output $(\mathcal{S}^*, \subseteq)$ in Algorithm 4.1 is a finite super-atomic lattice in $\mathcal{L}(n)$ by three steps as below.

(B1). It is easy to see that $(\mathcal{S}^*, \subseteq)$ has a minimum element \emptyset and a maximum element $\{1, 2, \dots, n\}$, respectively.

(B2). If $S \in \mathcal{S}^* - \mathcal{S}_1 \cup \mathcal{S}_0$ then there exist two elements $\{i\}, \{j\} \in \mathcal{S}_1$ such that $S = \{i\} \vee \{j\}$.

Suppose $S \in \mathcal{S}^* - \mathcal{S}_1 \cup \mathcal{S}_0$, then there exists $t \in \{2, 3, \dots, n\}$ such that $S \in \mathcal{S}_t$, and

$$\delta(S) = \{i_S, j_S\} \text{ with } \delta(S) \not\subseteq T \quad (12)$$

for any $T \in \mathcal{S}_t - \{S\}$ by Algorithm 4.1. Set

$$\mathcal{D} = \{D \in \mathcal{S}^* : \delta(S) \subseteq D\} \text{ and } \mathcal{D}_* = \{D : D \text{ is a minimal element of } \mathcal{D}\}. \quad (13)$$

We note that if $D \in \mathcal{D}_*$ then $D \neq T$ for any $T \in \mathcal{S}_t - \{S\}$ by (12) and (13). We claim that $D \notin \mathcal{S}_u$ for any integer u with $1 \leq u < t$. Indeed, if $D \in \mathcal{S}_u$, then $\delta(S) \subseteq D \subseteq S$ by Algorithm 4.1 and equation (12). Clearly, $D \neq S$ since $D \in \mathcal{S}_u$, $S \in \mathcal{S}_t$ and $t \neq u$. Thus $D \subsetneq S$. This implies that $D \subseteq S - \{i_S\}$ or $D \subseteq S - \{j_S\}$, contrary to $\delta(S) \subseteq D$.

Below, we assume that $D \in \mathcal{S}_v$ where $n \geq v \geq t$. We shall prove that $v = t$. Assume that $n \geq v > t$, then there exist $R \in \mathcal{S}_v$ such that

$$R \supsetneq S \quad (14)$$

by Algorithm 4.1. There are two cases.

Case (1). If $D = R$ then $D = R \supsetneq S$, a contradiction since $D \in \mathcal{D}_*$ and $S \in \mathcal{D}$.

Case (2). If $D \neq R$ then we must have that $\delta(D) = \delta(S)$. Otherwise, we have either $\delta(S) \subseteq D - \{i_D\} \subsetneq D$ or $\delta(S) \subseteq D - \{j_D\} \subsetneq D$, contrary to $D \in \mathcal{D}_*$. Hence, $\delta(D) = \delta(S) \subseteq S \subseteq R$, contrary to $\delta(D) \not\subseteq R$ since $R \neq D$ and both R and D in \mathcal{S}_v (see (12)).

Cases (1) and (2) imply that $v = t$. Therefore, $D = S$ by formulas (12) and (13), which means that \mathcal{D}_* contains exactly one element, i.e., \mathcal{D} has the least element S and $S = \bigvee \delta(S) = \{i_S\} \vee \{j_S\}$.

(B3). If $S_1, S_2 \in \mathcal{S}^*$ then $S_1 \vee S_2$ exists in \mathcal{S}^* .

We note that if S_1 and S_2 are comparable then $S_1 \vee S_2 = S_1$ or $S_1 \vee S_2 = S_2$. Next, we suppose that $S_1 \parallel S_2$. Then both S_1 and S_2 are not in \mathcal{S}_0 . We come to exactly the following three cases.

Case (i). Both $S_1 = \{i\}$ and $S_2 = \{j\}$ in \mathcal{S}_1 with $i \neq j$. We set

$$M = \{S \in \mathcal{S}^* : \{i, j\} \subseteq S\} \text{ and } M_* = \{S : S \text{ is a minimal element of } M\}. \quad (15)$$

We note that $M \neq \emptyset$ since $\{1, 2, \dots, n\} \in M$, and then $M_* \neq \emptyset$. Let S be an element in M_* . Then $S \in \mathcal{S}^* - \mathcal{S}_1 \cup \mathcal{S}_0$. Let $S \in \mathcal{S}_w$ for some $w \in \{2, 3, \dots, n\}$. Thus by (B2) and its proof, we have that $S = \bigvee \delta(S) = \{i_S\} \vee \{j_S\}$.

If $\{i, j\} \neq \delta(S)$ then $\{i, j\} \subseteq S - \{i_S\} \in \mathcal{S}^*$ or $\{i, j\} \subseteq S - \{j_S\} \in \mathcal{S}^*$ by Algorithm 4.1, contrary to the fact that $S \in M_*$. Therefore, $\{i, j\} = \delta(S)$, and then $S_1 \vee S_2 = \{i\} \vee \{j\} = S \in \mathcal{S}^*$.

Case (ii). $S_1 = \{i\} \in \mathcal{S}_1$ and $S_2 \in \mathcal{S}^* - \mathcal{S}_1 \cup \mathcal{S}_0$ with $i \notin S_2$. Let $S_2 \in \mathcal{S}_l$ for some $l \in \{2, 3, \dots, n\}$. Then by (B2) and its proof, we have that $S_2 = \bigvee \delta(S_2) = \{i_{S_2}\} \vee \{j_{S_2}\}$. Suppose that $S_1 \vee S_2$ does not exist in \mathcal{S}^* , then there exist two different minimal elements S_a and S_b in \mathcal{S}^* such that $S_1 \cup S_2 \subseteq S_a$ and $S_1 \cup S_2 \subseteq S_b$. Clearly, $S_a \parallel S_b$.

We claim that

$$\delta(S_a) \subseteq \{i, i_{S_2}, j_{S_2}\} \text{ and } \delta(S_a) \neq \delta(S_2). \quad (16)$$

Suppose that $\delta(S_a) \not\subseteq \{i, i_{S_2}, j_{S_2}\}$. Then $\{i, i_{S_2}, j_{S_2}\} \subseteq S_a - \{i_{S_a}\} \in \mathcal{S}^*$ or $\{i, i_{S_2}, j_{S_2}\} \subseteq S_a - \{j_{S_a}\} \in \mathcal{S}^*$ by Algorithm 4.1. If $\{i, i_{S_2}, j_{S_2}\} \subseteq S_a - \{i_{S_a}\}$ then $S_1 \cup S_2 \subseteq S_a - \{i_{S_a}\} \subsetneq S_a$, contrary to that S_a is minimal in \mathcal{S}^* with $S_1 \cup S_2 \subseteq S_a$. Similarly, we can prove that $\{i, i_{S_2}, j_{S_2}\} \subseteq S_a - \{j_{S_a}\} \in \mathcal{S}^*$ will result a contradiction. Next, we shall prove that $\delta(S_a) \neq \delta(S_2)$. Indeed, if $\delta(S_a) = \delta(S_2)$, then $S_a = \bigvee \delta(S_a) = S_2$ which implies $i \in S_2$, a contradiction.

Arguing as formula (16), we have

$$\delta(S_b) \subseteq \{i, i_{S_2}, j_{S_2}\} \text{ and } \delta(S_b) \neq \delta(S_2). \quad (17)$$

Formula (16) implies that $\delta(S_a)$ equals to $\{i, i_{S_2}\}$ or $\{i, j_{S_2}\}$, and formula (17) results that $\delta(S_b)$ equals to $\{i, i_{S_2}\}$ or $\{i, j_{S_2}\}$. If $\{i, i_{S_2}\} = \delta(S_a)$ then $\{i, j_{S_2}\} = \delta(S_b)$. Otherwise, we have that $\delta(S_a) = \{i, i_{S_2}\} = \delta(S_b)$, and which implies that $S_a = \bigvee \delta(S_a) = \bigvee \delta(S_b) = S_b$, a contradiction. On the other hand, $\{i, j_{S_2}\} \subseteq S_a - \{i_{S_2}\} \in \mathcal{S}^*$ which implies $S_b = \bigvee \delta(S_b) = \{i\} \vee \{j_{S_2}\} \subseteq S_a - \{i_{S_2}\} \subsetneq S_a$, contrary to $S_a \parallel S_b$. Similarly, we can prove that $\{i, j_{S_2}\} = \delta(S_a)$ will deduce a contradiction.

Therefore, $S_1 \vee S_2$ exists in \mathcal{S}^* .

Case (iii). Both S_1 and S_2 are in $\mathcal{S}^* - \mathcal{S}_1 \cup \mathcal{S}_0$ and $S_1 \parallel S_2$. If $\delta(S_1) \subseteq S_2$ then $\bigvee \delta(S_1) = S_1 \subseteq S_2$ by (B2), contrary to $S_1 \parallel S_2$. Thus $\delta(S_1) \not\subseteq S_2$. Similarly, we have $\delta(S_2) \not\subseteq S_1$.

Next, we assume that $S_1 \vee S_2$ does not exist in \mathcal{S}^* . Then there exist two different minimal elements C_1 and C_2 in \mathcal{S}^* such that $C_1 \supseteq S_1 \cup S_2$ and $C_2 \supseteq S_1 \cup S_2$. Clearly, $C_1 \parallel C_2$. We claim that

$$\delta(C_1) \subseteq \delta(S_1) \cup \delta(S_2). \quad (18)$$

Indeed, if $i_{C_1} \notin \delta(S_1) \cup \delta(S_2)$ then $\delta(S_1) \cup \delta(S_2) \subseteq C_1 - \{i_{C_1}\} \in \mathcal{S}^*$ by Algorithm 4.1. Thus $C_1 \supsetneq C_1 - \{i_{C_1}\} \supseteq S_1 \cup S_2$, contrary to the fact that C_1 is minimal. Therefore, $i_{C_1} \in \delta(S_1) \cup \delta(S_2)$. Similarly, we have $j_{C_1} \in \delta(S_1) \cup \delta(S_2)$.

Using (18), we know that $\delta(C_1)$ equals to one of four sets $\{i_{S_1}, i_{S_2}\}$, $\{i_{S_1}, j_{S_2}\}$, $\{j_{S_1}, i_{S_2}\}$ and $\{j_{S_1}, j_{S_2}\}$. Similarly, we can prove that $\delta(C_2)$ equals to one of four sets $\{i_{S_1}, i_{S_2}\}$, $\{i_{S_1}, j_{S_2}\}$, $\{j_{S_1}, i_{S_2}\}$ and $\{j_{S_1}, j_{S_2}\}$. We note that $\delta(C_1) \neq \delta(C_2)$ by the proof of cases (ii). Now, we suppose that $\delta(C_1) = \{i_{S_1}, i_{S_2}\}$. Then $C_1 - \{i_{S_1}\}, C_1 - \{i_{S_2}\} \in \mathcal{S}^*$ by Algorithm 4.1. If $\delta(C_2) = \{i_{S_1}, j_{S_2}\}$ then $C_2 = \{i_{S_1}\} \vee \{j_{S_2}\} \subseteq C_1 - \{i_{S_2}\}$, contrary to $C_1 \parallel C_2$. Similarly, we can prove that all the other cases will deduce a contradiction. Therefore, $S_1 \vee S_2$ exists in \mathcal{S}^* .

(B4). $(\mathcal{S}^*, \subseteq)$ is a finite super-atomic lattice.

By (B1), (B2), (B3) and Definitions 2.2 and 2.3, we know that the partially ordered set $(\mathcal{S}^*, \subseteq)$ is a finite atomic lattice. Next, we shall prove that the lattice $(\mathcal{S}^*, \subseteq)$ is super-atomic.

Suppose $S \in \mathcal{S}^* - \mathcal{S}_1 \cup \mathcal{S}_0$ and $T \in B_S$. It is easy to see that $\bigvee T = S$. If $\{i_S\} \notin T$ then $\bigcup T \subseteq S - \{i_S\} \in \mathcal{S}^*$, which implies $\bigvee T \subseteq S - \{i_S\}$, contrary to $\bigvee T = S$. Thus $\{i_S\} \in T$. Similarly, we have $\{j_S\} \in T$. Hence $\{\{i_S\}, \{j_S\}\} \subseteq T$. Again, by (B2) and its proof we know that $\{i_S\} \vee \{j_S\} = S$. Therefore, the lattice $(\mathcal{S}^*, \subseteq)$ is super-atomic by Definition 4.1.

We finally prove that every finite super-atomic lattice in $\mathcal{L}(n)$ can be constructed by Algorithm 4.1.

Note that

$$|L| = C_n^2 + (n+1) \quad (19)$$

for any finite super-atomic lattice $L \in \mathcal{L}(n)$ by Definition 4.1. Let (\mathcal{S}, \subseteq) be a finite super-atomic lattice in $\mathcal{L}(n)$. Then there exists a set \mathcal{S}^* such that $\mathcal{S} \subseteq \mathcal{S}^*$ by Algorithm 4.1 and Lemma 4.1 (In fact, there exist two elements $\{i\}, \{j\} \in \mathcal{S}$ such that $\{i\} \vee \{j\} = S$ for any $S \in \mathcal{S}$ with $|S| \geq 2$ since (\mathcal{S}, \subseteq) is a finite super-atomic lattice. Then we take $\delta(S) = \{i, j\}$ in Step 3. In this way, the final output \mathcal{S}^* of Algorithm 4.1 must satisfy $\mathcal{S} \subseteq \mathcal{S}^*$ by Lemma 4.1). If $\mathcal{S} \subsetneq \mathcal{S}^*$ then $|\mathcal{S}| < |\mathcal{S}^*|$. However, by (19) we know that $|\mathcal{S}| = C_n^2 + (n+1) = |\mathcal{S}^*|$ since $(\mathcal{S}^*, \subseteq)$ is super-atomic, a contradiction. Therefore, $\mathcal{S} = \mathcal{S}^*$. \square

The following two examples will illustrate Algorithm 4.1.

Example 4.1 Let $n = 3$. Then by Algorithm 4.1 we have three super-atomic lattices in $\mathcal{L}(n)$ as follows.

$$\mathcal{S}_{P_1} = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1, 2, 3\}, \{1, 2\}, \{1, 3\}\},$$

$$\mathcal{S}_{P_2} = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1, 2, 3\}, \{1, 2\}, \{2, 3\}\},$$

$$\mathcal{S}_{P_3} = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1, 2, 3\}, \{1, 3\}, \{2, 3\}\}.$$

It is easy to see that $(\mathcal{S}_{P_1}, \subseteq)$, $(\mathcal{S}_{P_2}, \subseteq)$ and $(\mathcal{S}_{P_3}, \subseteq)$ are all the finite super-atomic lattices in $\mathcal{L}(n)$.

Mapes proved the following lemma.

Lemma 4.2 ([9], **Theorem 4.3.2 and Proposition 4.3.1**) *With the partial order \leq , $\mathcal{L}(n)$ is a finite atomic lattice with $2^n - n - 2$ atoms, and if $P \geq Q$ in $\mathcal{L}(n)$ then P covers Q if and only if $|P| = |Q| + 1$.*

Based on Lemma 4.2, we have the following statement:

Theorem 4.3 *Let $P, Q \in \mathcal{L}(n)$ and $\text{atoms}(P) = \text{atoms}(Q)$. If $P \succ Q$, then the element in P corresponding to T is meet-irreducible in P where $T \in \mathcal{S}_P - \mathcal{S}_Q$.*

Proof. The condition $P \succ Q$ and $\text{atoms}(P) = \text{atoms}(Q)$ implies that $\mathcal{S}_P \supsetneq \mathcal{S}_Q$, and by Lemma 4.2 we have that $|\mathcal{S}_P - \mathcal{S}_Q| = 1$. Thus $\{T\} = \mathcal{S}_P - \mathcal{S}_Q$.

If the element in P corresponding to T is not meet-irreducible in P , then there exist two different elements $S_1, S_2 \in \mathcal{S}_P$ such that $S_1 \succ T$ and $S_2 \succ T$ in lattice \mathcal{S}_P . We note that $S_1 \in \mathcal{S}_Q$ and $S_2 \in \mathcal{S}_Q$. We claim that $\bigvee_{t \in T} \{t\} = S_1$ in lattice \mathcal{S}_Q . Otherwise, we have $\bigvee_{t \in T} \{t\} = R$ for some $R \in \mathcal{S}_Q$ with $R \subsetneq S_1$, and then $T \subseteq R \in \mathcal{S}_P$. Clearly, $T \neq R$ since $T \notin \mathcal{S}_Q$. This implies that $T \subsetneq R \subsetneq S_1$ in lattice \mathcal{S}_P , contrary to $S_1 \succ T$ in lattice \mathcal{S}_P . Therefore, $\bigvee_{t \in T} \{t\} = S_1$ in lattice \mathcal{S}_Q . Similarly, we can prove that $\bigvee_{t \in T} \{t\} = S_2$ in lattice \mathcal{S}_Q . Therefore, $S_1 = S_2$, contrary to $S_1 \neq S_2$. \square

5 Specific labelings

In [9], there are three specific coordinatizations, i.e., Minimal Squarefree, Minimal Depolarized and Greedy, one can see that all of them are based on the labeling described as in Lemma 2.2. In this section, we shall give a kind of labelings on a lattice P which does not satisfy the conditions of Lemma 2.2, and show the conditions that our labeling is either a coordinatization or a weak coordinatization.

Let $P \in \mathcal{L}(n)$ with $\text{atoms}(P) = \{a_1, a_2, \dots, a_n\}$. We define a labeling \mathcal{C} of P as that $\mathcal{C} = \{m_p : p \in P - \{0\}\}$ where

$$m_p = \prod_{a_i \in \text{supp}(p)} a_i \quad (20)$$

in which every a_i means both atom in P and variable in labeling \mathcal{C} .

In what follows, let $[a, b] = \{p \in P : a \leq p \leq b\}$ and $N([a, b]) = |[a, b]|$ for the purposes of convenience.

Theorem 5.1 *Let $P \in \mathcal{L}(n)$. If for any $p \in P - \text{atoms}(P) - \{0\}$, there exist two elements $a_i, a_j \in \text{supp}(p)$ such that $p = a_i \vee a_j$ satisfying that $N([a_r \vee a_k, 1]) < N([p, 1])$ for a fixed element r in $\{i, j\}$ and any $a_k \in \text{atoms}(P) - \text{supp}(p)$, then the labeling \mathcal{C} of P as defined by (20) is a weak coordinatization.*

Proof. For $b \in P$, define $g : P \longrightarrow LCM(I_{P,\mathcal{C}})$ to be the map such that

$$g(b) = \text{lcm}\{\Delta(u) : u \in \text{supp}(b)\}.$$

The main part is to show that g is an isomorphism of lattices. By the proof of Theorem 3.1, we know that the map g is meet-preserving, join-preserving and surjection. Thus, we only need to show that g is injective. The proof will be split into two parts.

(*) Let u, v be two atoms of P . Then $u \mid \Delta(v)$ if and only if $v \in \text{atoms}(P) - \{u\}$.

Suppose that $u \mid \Delta(v)$. By the definition of labeling \mathcal{C} , we know that $u \mid m_p$ if and only if $p \geq u$. Thus $u \nmid x(u)$ by the definition of $x(u)$. This means $u \nmid \Delta(u)$ since $\Delta(u) \mid x(u)$. Hence $u \neq v$, i.e., $v \in \text{atoms}(P) - \{u\}$.

Conversely, for any $q \in \text{atoms}(P) - \{u\}$, $u \in [q]^c$ implies $u \mid x(q)$ by equation (1). Since $u \neq \bigvee F \geq v$ for any $F \in \bigcup_{w \geq v} B_w$, there exists an atom $z \in \text{atoms}(P) - \{u\}$ such that $z \in F$. Thus $u \mid \text{lcm}\{x(f) : f \in F\}$ for any $F \in \bigcup_{w \geq v} B_w$. This together with equation (4) implies that $u \mid \Delta(v)$.

(**) The map g is injective.

Clearly, if $0 \in \{a, b\}$ and $g(a) = g(b)$ then $g(a) = g(b) = g(0) = 1$, which implies that $a = b = 0$. Next, let $a, b \in P - \{0\}$ be such that $g(a) = g(b)$. Then we need to show $a = b$.

Suppose that $b \not\leq a$. In this case, we have either $a \in \text{atoms}(P)$ or $a \in P - \text{atoms}(P) - \{0\}$. If $a \in \text{atoms}(P)$, then $\text{supp}(b) - \text{supp}(a) \neq \emptyset$ and $\text{supp}(a) = \{a\}$. By statement (*), we know that $a \nmid \Delta(a)$ and $a \mid \Delta(c)$ for any $c \in \text{supp}(b) - \text{supp}(a)$. Hence $a \mid g(b)$ and $a \nmid g(a)$, contrary to $g(a) = g(b)$.

Now, assume that $a \in P - \text{atoms}(P) - \{0\}$. Then by the hypothesis of the theorem, there exist two elements $a_i, a_j \in \text{supp}(a)$ such that $a = a_i \vee a_j$ satisfying $N([a_j \vee a_k, 1]) < N([a, 1])$ for any $a_k \in \text{atoms}(P) - \text{supp}(a)$. Let $a_j^{n_k}$ be the highest power of a_j dividing $x(a_k)$ and $a_j^{n_i}$ be the highest power of a_j dividing $x(a_i)$. Clearly, by (1) we have that

$$x(a_k) = \prod_{q \in [a_k]^c} m_q = \prod_{q_1 \in [a_k]^c \cap [a_j]} m_{q_1} * \prod_{q_2 \in [a_k]^c \cap [a_j]^c} m_{q_2}$$

in which $a_j \nmid m_{q_2}$, $a_j \mid m_{q_1}$ by the definition of labeling \mathcal{C} . Thus $n_k = |[a_k]^c \cap [a_j]|$ where $[a_k]^c \cap [a_j] = [a_j, 1] - [a_j \vee a_k, 1]$. Hence $n_k = N([a_j, 1]) - N([a_j \vee a_k, 1])$. Similarly, we have that $n_i = N([a_j, 1]) - N([a_j \vee a_i, 1])$. Therefore,

$$n_k - n_i = N([a_j \vee a_i, 1]) - N([a_j \vee a_k, 1]) = N([a, 1]) - N([a_j \vee a_k, 1]) \geq 1. \quad (21)$$

Next, let $r \geq a_k$. If $T \in B_r$ then there must exist an element $a_t \in T$ such that $a_t \notin \text{supp}(a)$. Otherwise, we have that $T \subseteq \text{supp}(a)$, which yields $a_k \leq r = \bigvee T \leq \bigvee \text{supp}(a) = a$, contrary to $a_k \notin \text{supp}(a)$. It follows from (21) that

$$a_j^{n_i+1} \mid \text{lcm}\{x(a_t) : a_t \in T\} \quad (22)$$

for any $T \in B_r$. Below, assume that $a_j^{m_k}$ is the highest power of a_j dividing $\Delta(a_k)$ and $a_j^{m_i}$ is the highest power of a_j dividing $\Delta(a_i)$. Thus $m_k \geq n_i + 1$ by formulas (22) and (4). Hence

$$m_k > n_i \geq m_i \quad (23)$$

for any $a_k \in \text{atoms}(P) - \text{supp}(a)$ since $\Delta(a_i) \mid x(a_i)$.

Again, clearly $\text{supp}(b) - \text{supp}(a) \neq \emptyset$ since $b \not\leq a$, i.e., there exists an element $a_s \in \text{supp}(b) - \text{supp}(a)$ such that $a_s \vee a_e = b$ for some $a_e \in \text{atoms}(P)$. This follows that $g(b) = \text{lcm}\{\Delta(a_s), \Delta(a_e)\}$ since the map g is join-preserving. Now, let a_j^m be the highest power of a_j dividing $g(b)$. Then $m \geq m_s$. Using formula (23), we know that $m_s > m_i$ since $a_s \in \text{supp}(b) - \text{supp}(a) \subseteq \text{atoms}(P) - \text{supp}(a)$, which together with $m \geq m_s$ follows that $m > m_i$.

On the other hand, we have that $g(a) = \text{lcm}\{\Delta(a_i), \Delta(a_j)\}$ since $a_i \vee a_j = a$ and the map g is join-preserving, and $a_j \nmid \Delta(a_j)$ by statement (*). Thus $a_j^{m_i}$ is the highest power of a_j dividing $g(a)$. Hence, we finally have that $g(b) \nmid g(a)$ since $m > m_i$, contrary to $g(a) = g(b)$.

Therefore, the assumption of $b \not\leq a$ will deduce a contradiction. Hence $b \leq a$.

Similarly, we can prove that $a \leq b$, which together with $b \leq a$ implies that $a = b$ finally. \square

Remark 5.1 The labeling \mathcal{C} of P as defined by (20) needs not to satisfy the conditions (C1) and (C2) of Theorem 3.1 generally. For example, consider the lattice shown as Fig.9.

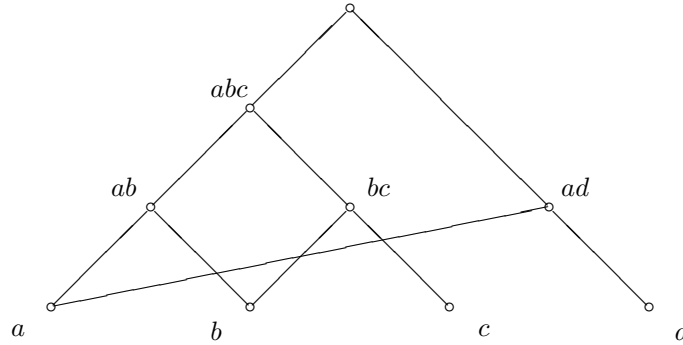


Fig.9. The lattice P with a labeling \mathcal{C}

Clearly, the lattice P satisfies the conditions of Theorem 5.1, and its labeling \mathcal{C} yields that $I_{P,\mathcal{C}} = \{b^2c^2d, a^2cd^2, a^3b^2d^2, a^3b^4c^3\}$ with $\text{LCM}(I_{P,\mathcal{C}}) \cong P$. Obviously, the labeling \mathcal{C} does not satisfy the conditions (C1) and (C2) of Theorem 3.1.

Theorem 5.2 *Let P be a finite super-atomic lattice and $p \in P$. Then the labeling \mathcal{C} of P as defined by (20) is a strong coordinatization if and only if either $N([a_i \vee a_k, 1]) \leq N([a_r \vee a_k, 1])$ or $N([a_j \vee a_k, 1]) \leq N([a_r \vee a_k, 1])$ for any $a_k, a_r \in \text{supp}(p)$ where $p = a_i \vee a_j$ with $a_i, a_j \in \text{supp}(p)$.*

Proof. Let g be an isomorphic map from P to $\text{LCM}(C_{P,\mathcal{C}})$ satisfying $g(a) = x(a)$ for each $a \in \text{atoms}(P)$ since \mathcal{C} is a strong coordinatization. Suppose there exist two elements $a_k, a_r \in \text{supp}(p)$ such that

$$N([a_i \vee a_k, 1]) > N([a_r \vee a_k, 1]) \text{ and } N([a_j \vee a_k, 1]) > N([a_r \vee a_k, 1])$$

where $p = a_i \vee a_j$ with $a_i, a_j \in \text{supp}(p)$. Let $a_k^{n_r}$ be the highest power of a_k dividing $x(a_r)$, $a_k^{n_i}$ the highest power of a_k dividing $x(a_i)$ and $a_k^{n_j}$ the highest power of a_k dividing $x(a_j)$. Then by the proof of Theorem 5.1, we have $n_r > n_i$ and $n_r > n_j$. Thus $a_k^{n_r} \nmid \text{lcm}\{x(a_i), x(a_j)\}$, i.e., $x(a_r) \nmid \text{lcm}\{x(a_i), x(a_j)\}$. Again, $g(p) = \text{lcm}\{x(a_i), x(a_j)\}$. Thus

$x(a_r) \nmid g(p)$. However, $a_r \leq p$, which means $g(a_r) = x(a_r) \leq g(p)$, i.e., $x(a_r) \mid g(p)$, a contradiction.

Conversely, suppose that either $N([a_i \vee a_k, 1]) \leq N([a_r \vee a_k, 1])$ or $N([a_j \vee a_k, 1]) \leq N([a_r \vee a_k, 1])$ for any $a_k, a_r \in \text{supp}(p)$ where $p = a_i \vee a_j$ with $a_i, a_j \in \text{supp}(p)$.

In what follows, we first prove that $\triangle(a) = x(a)$ for any atoms a in P . The proof will be completed by two parts.

(D1). We shall prove that

$$x(a_s) \mid \text{lcm}\{x(a_i), x(a_j)\} \text{ if } a_s \in \text{supp}(p) - \{a_i, a_j\}. \quad (24)$$

Let $a_t \in \text{atoms}(P)$. If $a_s \in \text{supp}(p) - \{a_i, a_j\}$ then $a_i \vee a_s < p$ and $a_j \vee a_s < p$ since P is finite super-atomic. Let $a_t^{n_s}$ be the highest power of a_t dividing $x(a_s)$, $a_t^{n_i}$ the highest power of a_t dividing $x(a_i)$ and $a_t^{n_j}$ the highest power of a_t dividing $x(a_j)$. Now, if $a_t \neq a_s$, then there are two cases.

Case (1*). Suppose $a_t \notin \text{supp}(p)$. We first prove that either $a_i \vee a_j \vee a_t \vee a_s = a_t \vee a_i$ or $a_i \vee a_j \vee a_t \vee a_s = a_t \vee a_j$. Indeed, it is easy to see that $a_i \vee a_j \neq a_i \vee a_j \vee a_t \vee a_s$, $a_i \vee a_s \neq a_i \vee a_j \vee a_t \vee a_s$ and $a_j \vee a_s \neq a_i \vee a_j \vee a_t \vee a_s$ since $a_s \in \text{supp}(p)$ and $a_t \notin \text{supp}(p)$. Now, assume that $a_s \vee a_t = a_i \vee a_j \vee a_t \vee a_s$. Then by Definition 4.1, $a_i \vee a_j \vee a_t \neq a_i \vee a_j \vee a_t \vee a_s$, a contradiction (since $a_s \in \text{supp}(p)$ and $a_i \vee a_j = p$ deduces that $a_i \vee a_j \vee a_t \vee a_s = a_i \vee a_j \vee a_t$). Therefore, from Definition 4.1 we know that either $a_i \vee a_j \vee a_t \vee a_s = a_t \vee a_i$ or $a_i \vee a_j \vee a_t \vee a_s = a_t \vee a_j$.

Now, if $a_i \vee a_j \vee a_t \vee a_s = a_t \vee a_i$ then $a_s \vee a_t < a_i \vee a_j \vee a_t \vee a_s$ since $a_s \vee a_t \neq a_i \vee a_j \vee a_t \vee a_s$. Thus $N([a_s \vee a_t, 1]) > N([a_i \vee a_t, 1])$. By the proof of Theorem 5.1, we know that $a_t^{n_s} \mid a_t^{n_i}$, i.e., $a_t^{n_s} \mid \text{lcm}\{x(a_i), x(a_j)\}$. Similarly, we can prove that $a_t^{n_s} \mid \text{lcm}\{x(a_i), x(a_j)\}$ if $a_i \vee a_j \vee a_t \vee a_s = a_t \vee a_j$. Therefore, we always have that $a_t^{n_s} \mid \text{lcm}\{x(a_i), x(a_j)\}$ in the case of $a_t \notin \text{supp}(p)$.

Case (2*). Suppose $a_t \in \text{supp}(p)$. Then either $N([a_i \vee a_t, 1]) \leq N([a_s \vee a_t, 1])$ or $N([a_j \vee a_t, 1]) \leq N([a_s \vee a_t, 1])$. If $N([a_i \vee a_t, 1]) \leq N([a_s \vee a_t, 1])$ then $n_s \leq n_i$, i.e., $a_t^{n_s} \mid \text{lcm}\{x(a_i), x(a_j)\}$. Similarly, we can prove that if $N([a_j \vee a_t, 1]) \leq N([a_s \vee a_t, 1])$ then $n_s \leq n_j$, i.e., $a_t^{n_s} \mid \text{lcm}\{x(a_i), x(a_j)\}$. Therefore, we always have that $a_t^{n_s} \mid \text{lcm}\{x(a_i), x(a_j)\}$ in the case of $a_t \in \text{supp}(p)$.

If $a_s = a_t$ then clearly $n_s = 0$, which together with Cases (1*) and (2*) means that

$$x(a_s) \mid \text{lcm}\{x(a_i), x(a_j)\} \text{ if } a_s \in \text{supp}(p) - \{a_i, a_j\}.$$

(D2). We shall prove that

$$\triangle(a) = x(a) \quad (25)$$

for any atoms a in P .

Let $q \in P$ and $q \geq a$. If $q = a$ then $B_q = \{\{a\}\}$. Clearly, we have that

$$x(a) \mid \text{lcm}\{x(r) : r \in T\} \quad (26)$$

for any $T \in B_q$. If $q > a$ then there exist two elements $q_1, q_2 \in \text{supp}(q)$ such that $q_1 \vee q_2 = q$ by Definition 4.1. Thus $q_1, q_2 \in T$ for any $T \in B_q$. Using (24), we have $x(a) \mid \text{lcm}\{x(q_1), x(q_2)\}$, then

$$x(a) \mid \text{lcm}\{x(r) : r \in T\} \quad (27)$$

for any $T \in B_q$. Formulas (26) and (27) mean that $x(a) \mid \text{lcm}\{x(r) : r \in T\}$ for any $T \in B_q$ if $q \geq a$. Thus $x(a) \mid \Delta(a)$ by Definition 3.2. Clearly, $\Delta(a) \mid x(a)$ and therefore $x(a) = \Delta(a)$, and which implies $I_{P,\mathcal{C}} = C_{P,\mathcal{C}}$.

Next, we shall prove that the lcm-lattice of $I_{P,\mathcal{C}}$ is isomorphic to P according Lemma 3.1, i.e., prove the labeling \mathcal{C} is a weak coordinatization.

For $q \in P$, define $g : P \longrightarrow LCM(I_{P,\mathcal{C}}) = LCM(C_{P,\mathcal{C}})$ to be the map such that

$$g(q) = \text{lcm}\{\Delta(w) : w \in \text{supp}(q)\} = \text{lcm}\{x(w) : w \in \text{supp}(q)\}.$$

One can check that g is meet-preserving, join-preserving and surjection by the proof of Theorem 3.1. Hence we only need to prove that g is injective. Clearly, if $0 \in \{u, v\}$ and $g(u) = g(v)$ then $g(u) = g(v) = g(0) = 1$, which implies that $u = v = 0$. Next, suppose $g(u) = g(v)$ for $u, v \in P - \{0\}$. If $v \not\leq u$ then $\text{supp}(v) - \text{supp}(u) \neq \emptyset$. Take an element $c_t \in \text{supp}(v) - \text{supp}(u)$. There are two cases as below.

Case (k1). If $u \in \text{atoms}(P)$ then $\text{supp}(u) = \{u\}$. Thus by statement (*) in the proof of Theorem 5.1, we know that $u \nmid \Delta(u)$ and $u \mid \Delta(c_t)$. Hence $u \mid g(v)$ and $u \nmid g(u)$, contrary to $g(u) = g(v)$.

Case (k2). If $u \in P - \text{atoms}(P) - \{0\}$, then there exists exactly two elements $c_i, c_j \in \text{supp}(u)$ such that $c_i \vee c_j = u$, which implies that

$$g(u) = \text{lcm}\{x(c_i), x(c_j)\}. \quad (28)$$

On the other hand, we have either $c_t \vee c_i = c_t \vee c_i \vee c_j$ or $c_t \vee c_j = c_t \vee c_i \vee c_j$ since P is super-atomic lattice. Assume that $c_t \vee c_i = c_t \vee c_i \vee c_j$. We note that $c_t \vee c_i > c_j \vee c_i$ since $c_t \notin \text{supp}(u)$. Then $N([c_e \vee c_i, 1]) > N([c_t \vee c_i, 1])$ for any $e \in \{i, j\}$. Let $c_i^{n_e}$ be the highest power of c_i dividing $x(c_e)$ and $c_i^{n_t}$ the highest power of c_i dividing $x(c_t)$. Then we have that $n_t > n_e$, which together with formula (28) yields $c_i^{n_t} \nmid g(u)$. However, $c_i^{n_t} \mid g(v)$ since $c_t \in \text{supp}(v)$ and $g(v) = \text{lcm}\{x(h) : h \in \text{supp}(v)\}$. Hence $g(v) \nmid g(u)$, contrary to $g(u) = g(v)$. If $c_t \vee c_j = c_t \vee c_i \vee c_j$, then with analogous proof to the case of $c_t \vee c_i = c_t \vee c_i \vee c_j$ one can deduce a contradiction.

Cases (k1) and (k2) tell us that the assumption of $v \not\leq u$ will yield a contradiction. Hence $v \leq u$.

Arguing as above, we can prove that $u \leq v$. Therefore, we finally have that $u = v$. \square

Using Theorem 5.2 we can determine whether the labeling, defined by (20), of a super-atomic lattice in $\mathcal{L}(n)$ is a coordinatization. As a conclusion of this section, we shall consider when the labeling, defined by (20), of a non-super-atomic lattice is also a coordinatization.

For any $T \in \mathcal{L}(n)$, we suppose that $\{a_1, a_2, \dots, a_n\} = \text{atoms}(T)$. Next we denote by \mathcal{C}_T the labeling of T defined by (20), that is, $m_c = \prod_{a_i \in \text{supp}(c)} a_i$ for any $c \in T - \{0\}$. Note that \mathcal{S}_T is the lattice corresponding to T (see Section 2). Then for any $C \in \mathcal{S}_T - \{\emptyset\}$, we have that $m_C = \prod_{a_i \in C} a_i$ and $\{\{a_1\}, \{a_2\}, \dots, \{a_n\}\} = \text{atoms}(\mathcal{S}_T)$ where C is the element in \mathcal{S}_T corresponding to c . Again, we denote by $x_T(\{a_i\})$ the monomials corresponding to \mathcal{S}_T defined by (1). Then we define $C_{\mathcal{S}_T, \mathcal{C}_T}$ as the ideal generated by monomials $x_T(\{a_i\})$ for each $\{a_i\} \in \text{atoms}(\mathcal{S}_T)$. We denote by $\Delta_T(\{a_i\})$ the monomials corresponding to \mathcal{S}_T defined by (4), and define $I_{\mathcal{S}_T, \mathcal{C}_T}$ as the ideal generated by monomials $\Delta_T(\{a_i\})$ for each $\{a_i\} \in \text{atoms}(\mathcal{S}_T)$. Then we have the following theorem.

Theorem 5.3 *Let \mathcal{S}_R be a super-atomic lattice in $\mathcal{L}(n)$, $\mathcal{S}_Q \prec \mathcal{S}_P \leq \mathcal{S}_R$ and \mathcal{C}_P a strong coordinatization. Then \mathcal{C}_Q is a strong coordinatization if and only if $\Delta_Q(\{a_k\}) = x_Q(\{a_k\})$ for any $k \in \{1, 2, \dots, n\}$.*

Proof. We only need to show the sufficiency of theorem since the necessity is obvious. According to Lemma 3.1, we just need to prove that \mathcal{C}_Q is a weak coordinatization, i.e., we need to prove that the lcm-lattice of $I_{\mathcal{S}_Q, \mathcal{C}_Q}$ is isomorphic to \mathcal{S}_Q . We first note that $I_{\mathcal{S}_Q, \mathcal{C}_Q} = C_{\mathcal{S}_Q, \mathcal{C}_Q}$ since $\Delta_Q(\{a_k\}) = x_Q(\{a_k\})$ for any $k \in \{1, 2, \dots, n\}$.

Define a map $h : \mathcal{S}_Q \rightarrow LCM(I_{\mathcal{S}_Q, \mathcal{C}_Q}) = LCM(C_{\mathcal{S}_Q, \mathcal{C}_Q})$ as

$$h(C) = \text{lcm}\{\Delta_Q(\{a_i\}) : a_i \in C\} = \text{lcm}\{x_Q(\{a_i\}) : a_i \in C\}$$

for any $C \in \mathcal{S}_Q$. By the proof of Theorem 3.1, one can check that h is meet-preserving, join-preserving and surjective. Now, we shall prove that h is injective.

For $C \in \mathcal{S}_P$, we define an isomorphic map $g : \mathcal{S}_P \rightarrow LCM(C_{\mathcal{S}_P, \mathcal{C}_P})$ such that

$$g(C) = \text{lcm}\{x_P(\{a_i\}) : a_i \in C\}$$

since \mathcal{C}_P is a strong coordinatization.

By Lemma 4.2, we know that $|\mathcal{S}_P| = |\mathcal{S}_Q| + 1$ since $\mathcal{S}_P \succ \mathcal{S}_Q$. Now, let $\{S\} = \mathcal{S}_P - \mathcal{S}_Q$. Then there exists exactly one element $T \in \mathcal{S}_P$ such that $T \succ S$ in \mathcal{S}_P by Theorem 4.3.

We note that if $a_j \in \{a_1, a_2, \dots, a_n\} - S$ then $S \notin [\{a_j\}]_P$ since $\{a_j\} \not\subseteq S$ in \mathcal{S}_P . Thus $[\{a_j\}]_P = [\{a_j\}]_Q$, which implies that

$$x_Q(\{a_j\}) = \prod_{C \in [\{a_j\}]_Q^c} m_C = \prod_{C \in \mathcal{S}_Q - [\{a_j\}]_Q} m_C = \frac{\prod_{C \in \mathcal{S}_P - [\{a_j\}]_P} m_C}{m_S} = \frac{x_P(\{a_j\})}{\prod_{a_i \in S} a_i}. \quad (29)$$

On the other hand, if $a_j \in S$ then $S \in [\{a_j\}]_P$ since $\{a_j\} \subseteq S$. Thus $[\{a_j\}]_P = [\{a_j\}]_Q \cup \{S\}$, which implies that

$$x_Q(\{a_j\}) = \prod_{C \in [\{a_j\}]_Q^c} m_C = \prod_{C \in \mathcal{S}_Q - [\{a_j\}]_Q} m_C = \prod_{C \in \mathcal{S}_P - [\{a_j\}]_P} m_C = x_P(\{a_j\}). \quad (30)$$

The following proof is completed by three parts.

(I) If $h(C_1) = h(D_1)$ and $C_1 \subseteq D_1$ in \mathcal{S}_Q then $C_1 = D_1$.

Suppose that $C_1 \subsetneq D_1$. Then there exists an element $C_2 \in \mathcal{S}_Q$ such that $C_1 \prec C_2 \subseteq D_1$ since \mathcal{S}_Q is finite, and

$$h(C_1) = h(C_2) \quad (31)$$

since h is meet-preserving.

Clearly, if $C_1 = \emptyset$ then $h(C_1) = 1 = h(D_1)$, and which implies that $C_1 = D_1$. Next, we suppose that $C_1 \in \mathcal{S}_Q - \{\emptyset\}$.

If $C_1 \in \text{atoms}(\mathcal{S}_Q)$, then let $C_1 = \{a_u\}$. Clearly, there exists an element $\{a_v\} \in \text{supp}(C_2)$ satisfying $\{a_v\} \neq \{a_u\}$ since $C_1 \prec C_2$. By statement (*) in the proof of Theorem

5.1, we know that $a_u \nmid \Delta_Q(\{a_u\})$ and $a_u \mid \Delta_Q(\{a_v\})$. Hence, $a_u \mid h(C_2)$ and $a_u \nmid h(C_1)$, contrary to formula (31).

If $C_1 \in \mathcal{S}_Q - \text{atoms}(\mathcal{S}_Q) - \{\emptyset\}$, then there exist two atoms $\{a_i\}, \{a_j\} \in \mathcal{S}_Q$ such that

$$C_1 = \{a_i\} \vee \{a_j\} \quad (32)$$

since \mathcal{S}_R is super-atomic and $\mathcal{S}_Q \leq \mathcal{S}_R$, and there exists an atom $\{a_k\} \in \mathcal{S}_Q$ such that

$$C_2 = \{a_i\} \vee \{a_j\} \vee \{a_k\} \quad (33)$$

since $C_1 \prec C_2$. Using formulas (31), (32) and (33), we know that

$$h(C_1) = \text{lcm}\{x_Q(\{a_i\}), x_Q(\{a_j\})\} = \text{lcm}\{x_Q(\{a_i\}), x_Q(\{a_j\}), x_Q(\{a_k\})\} = h(C_2). \quad (34)$$

Thus we shall distinguish the six types as follows.

Type 1. $a_i, a_j, a_k \in S$.

We claim that $C_1 \neq T$ in \mathcal{S}_Q . If $C_1 = T$ in \mathcal{S}_Q , then we have $\{a_i\} \vee \{a_j\} = S$ in \mathcal{S}_P since $a_i, a_j \in S$. Thus $\{a_i\} \vee \{a_j\} \vee \{a_k\} = S$ in \mathcal{S}_P since $a_k \in S$. By formulas (32) and (33), we know that $C_2 = T = C_1$ in \mathcal{S}_Q , a contradiction. Hence $C_1 \neq T$, and $C_1 \subsetneq T$ in \mathcal{S}_Q since $\{a_i, a_j\} \subseteq T$ in \mathcal{S}_Q . Therefore,

$$\{a_i\} \vee \{a_j\} = C_1 \subsetneq S \quad (35)$$

in \mathcal{S}_P .

Using formula (30), we have $x_Q(\{a_t\}) = x_P(\{a_t\})$ for any $t \in \{i, j, k\}$. Then

$$\text{lcm}\{x_P(\{a_i\}), x_P(\{a_j\})\} = \text{lcm}\{x_P(\{a_i\}), x_P(\{a_j\}), x_P(\{a_k\})\} \quad (36)$$

by formula (34).

If $C_2 = T$ in \mathcal{S}_Q then $\{a_i\} \vee \{a_j\} \vee \{a_k\} = S$ in \mathcal{S}_P since $a_i, a_j, a_k \in S$, which together with formulas (35) and (36) implies that $g(C_1) = g(S)$. However, $g(C_1) < g(S)$ since $C_1 \subsetneq S$ and the map g is isomorphic, a contradiction.

If $C_2 \neq T$ in \mathcal{S}_Q then $\{a_i\} \vee \{a_j\} \vee \{a_k\} = C_2$ in \mathcal{S}_P . By formulas (35) and (36), we know that $g(C_1) = g(C_2)$. However, $g(C_1) < g(C_2)$ since $C_1 \subsetneq C_2$ and the map g is isomorphic, a contradiction.

Type 2. $a_i, a_j, a_k \notin S$.

By formula (29), we have $x_P(\{a_t\}) = (\prod_{a_r \in S} a_r) * x_Q(\{a_t\})$ for any $t \in \{i, j, k\}$. Thus $h(C_1) * \prod_{a_r \in S} a_r = h(C_2) * \prod_{a_r \in S} a_r$ since $h(C_1) = h(C_2)$. By formula (34), we have the formula

$$\text{lcm}\{x_P(\{a_i\}), x_P(\{a_j\})\} = \text{lcm}\{x_P(\{a_i\}), x_P(\{a_j\}), x_P(\{a_k\})\}.$$

On the other hand, we have $\{a_i\} \vee \{a_j\} = C_1$ and $\{a_i\} \vee \{a_j\} \vee \{a_k\} = C_2$ in \mathcal{S}_P since $a_i, a_j, a_k \notin S$. Therefore, $g(C_1) = g(C_2)$, contrary to the fact that $g(C_1) < g(C_2)$.

Type 3. $a_i, a_j \notin S$ and $a_k \in S$.

By formulas (30) and (34), we have that

$$\text{lcm}\{x_Q(\{a_i\}), x_Q(\{a_j\})\} = \text{lcm}\{x_Q(\{a_i\}), x_Q(\{a_j\}), x_P(\{a_k\})\}.$$

Thus $x_P(\{a_k\}) \mid \text{lcm}\{x_Q(\{a_i\}), x_Q(\{a_j\})\}$. Similar to the proof of Type 2, we know that $\{a_i\} \vee \{a_j\} = C_1$, $\{a_i\} \vee \{a_j\} \vee \{a_k\} = C_2$ in \mathcal{S}_P and $x_P(\{a_t\}) = (\prod_{a_r \in S} a_r) * x_Q(\{a_t\})$ for any $t \in \{i, j\}$. Thus $x_P(\{a_k\}) \mid \text{lcm}\{x_P(\{a_i\}), x_P(\{a_j\})\}$, which implies that

$$\text{lcm}\{x_P(\{a_i\}), x_P(\{a_j\})\} = \text{lcm}\{x_P(\{a_i\}), x_P(\{a_j\}), x_P(\{a_k\})\}.$$

Therefore, $g(C_1) = g(C_2)$, contrary to the fact that $g(C_1) < g(C_2)$.

Type 4. $a_i \in S$, $a_j \notin S$ and $a_k \in S$.

Using (30) and (34), we have that

$$\text{lcm}\{x_P(\{a_i\}), x_Q(\{a_j\})\} = \text{lcm}\{x_P(\{a_i\}), x_Q(\{a_j\}), x_P(\{a_k\})\}.$$

Similar to the proof of Type 3, we have that $x_P(\{a_k\}) \mid \text{lcm}\{x_P(\{a_i\}), x_P(\{a_j\})\}$ and $g(C_1) = g(C_2)$ with $\{a_i\} \vee \{a_j\} = C_1$ and $\{a_i\} \vee \{a_j\} \vee \{a_k\} = C_2$ in \mathcal{S}_P , contrary to the fact that $g(C_1) < g(C_2)$.

Type 5. $a_i, a_j \in S$ and $a_k \notin S$.

Using (30) and (34), we have that

$$\text{lcm}\{x_P(\{a_i\}), x_P(\{a_j\})\} = \text{lcm}\{x_P(\{a_i\}), x_P(\{a_j\}), x_Q(\{a_k\})\}.$$

Then

$$x_Q(\{a_k\}) \mid \text{lcm}\{x_P(\{a_i\}), x_P(\{a_j\})\}. \quad (37)$$

Let $a_i^{m_{k_i}}$ be the highest power of a_i dividing $x_P(\{a_k\})$ and $a_i^{n_{k_i}}$ the highest power of a_i dividing $x_Q(\{a_k\})$. Using (29), we have $x_P(\{a_k\}) = (\prod_{a_r \in S} a_r) * x_Q(\{a_k\})$. Thus $n_{k_i} + 1 = m_{k_i}$ since $a_i \in S$. We note that $\{a_i\} \vee \{a_j\} \vee \{a_k\} = C_2$ in \mathcal{S}_P since $a_k \notin S$. Then

$$\{a_i\} \vee \{a_k\} = C_2 \text{ or } \{a_j\} \vee \{a_k\} = C_2$$

in \mathcal{S}_P since $\mathcal{S}_P \leq \mathcal{S}_R$ and \mathcal{S}_R is super-atomic. There are two subcases.

Subcase 1. If $C_1 = T$ in \mathcal{S}_Q then $\{a_i\} \vee \{a_j\} = S$ in \mathcal{S}_P . Thus $S \subsetneq C_1 = T \subsetneq C_2$ in \mathcal{S}_P .

Assume that $\{a_i\} \vee \{a_k\} = C_2$ in \mathcal{S}_P . Then we have that $N([\{a_i\} \vee \{a_j\}, 1]) \geq N([\{a_i\} \vee \{a_k\}, 1]) + 2$ since $S \subsetneq T \subsetneq C_2$ in \mathcal{S}_P . Let $a_i^{m_{j_i}}$ be the highest power of a_i dividing $x_P(\{a_j\})$. By the proof of Theorem 5.1, we have that $m_{k_i} \geq m_{j_i} + 2$. Thus $n_{k_i} \geq m_{j_i} + 1$ which implies that $x_Q(\{a_k\}) \nmid x_P(\{a_j\})$. By statement (*) in the proof of Theorem 5.1, we know that $a_i \nmid x_P(\{a_i\})$. Therefore, $x_Q(\{a_k\}) \nmid \text{lcm}\{x_P(\{a_i\}), x_P(\{a_j\})\}$, contrary to formula (37).

If $\{a_j\} \vee \{a_k\} = C_2$, then with analogous proof to the case of $\{a_i\} \vee \{a_k\} = C_2$ one may get a contradiction.

Subcase 2. If $C_1 \neq T$ in \mathcal{S}_Q then $\{a_i\} \vee \{a_j\} = C_1$ and $C_1 \subsetneq S$ in \mathcal{S}_P by the proof of Type 1.

Suppose that $\{a_i\} \vee \{a_k\} = C_2$ in \mathcal{S}_P . Then we have that $N([\{a_i\} \vee \{a_j\}, 1]) > N([\{a_i\} \vee \{a_k\}, 1])$ since $C_1 \subsetneq C_2$. We note that $C_2 \not\subseteq S$ since $a_k \notin S$. Thus we have that $N([\{a_i\} \vee \{a_j\}, 1]) \geq N([\{a_i\} \vee \{a_k\}, 1]) + 2$ since $C_1 \subsetneq S$ in \mathcal{S}_P . Similar to Subcase 1, one can prove that $x_Q(\{a_k\}) \nmid \text{lcm}\{x_P(\{a_i\}), x_P(\{a_j\})\}$, contrary to formula (37).

If $\{a_j\} \vee \{a_k\} = C_2$, then with analogous proof to the case of $\{a_i\} \vee \{a_k\} = C_2$ one may get a contradiction.

Type 6. $a_i \in S$ and $a_j, a_k \notin S$.

By (30) and (34), we have that

$$\text{lcm}\{x_P(\{a_i\}), x_Q(\{a_j\})\} = \text{lcm}\{x_P(\{a_i\}), x_Q(\{a_j\}), x_Q(\{a_k\})\}.$$

Thus

$$x_Q(\{a_k\}) \mid \text{lcm}\{x_P(\{a_i\}), x_Q(\{a_j\})\}. \quad (38)$$

Clearly, $\{a_i\} \vee \{a_j\} = C_1$ and $\{a_i\} \vee \{a_j\} \vee \{a_k\} = C_2$ in \mathcal{S}_P since $a_j, a_k \notin S$. By the proof of Type 5, we know that

$$\{a_i\} \vee \{a_k\} = C_2 \text{ or } \{a_j\} \vee \{a_k\} = C_2$$

in \mathcal{S}_P . There are two subcases.

Subcase (i). If $\{a_i\} \vee \{a_k\} = C_2$. By the proof of Type 5, we have that $N([\{a_i\} \vee \{a_j\}, 1]) > N([\{a_i\} \vee \{a_k\}, 1])$ in \mathcal{S}_P . Let $a_i^{m_{k_i}}$ be the highest power of a_i dividing $x_P(\{a_k\})$ and $a_i^{m_{j_i}}$ the highest power of a_i dividing $x_P(\{a_j\})$. Clearly, $m_{k_i} > m_{j_i}$, i.e., $x_P(\{a_k\}) \nmid x_P(\{a_j\})$. Using (29), we have

$$x_P(\{a_j\}) = \left(\prod_{a_r \in S} a_r\right) * x_Q(\{a_j\}) \text{ and } x_P(\{a_k\}) = \left(\prod_{a_r \in S} a_r\right) * x_Q(\{a_k\}).$$

Hence $x_Q(\{a_k\}) \nmid x_Q(\{a_j\})$. Again, $a_i \nmid x_P(\{a_i\})$ by statement (*) in the proof of Theorem 5.1. Thus $x_Q(\{a_k\}) \nmid \text{lcm}\{x_P(\{a_i\}), x_Q(\{a_j\})\}$, contrary to the formula (38).

Subcase (ii). If $\{a_j\} \vee \{a_k\} = C_2$. We note that $N([\{a_i\} \vee \{a_j\}, 1]) > N([\{a_j\} \vee \{a_k\}, 1])$ in \mathcal{S}_Q . Let $a_j^{n_{k_j}}$ be the highest power of a_j dividing $x_Q(\{a_k\})$ and $a_j^{n_{i_j}}$ the highest power of a_j dividing $x_Q(\{a_i\})$. Clearly, $n_{k_j} > n_{i_j}$. Again, we know that $a_j^{n_{i_j}}$ is the highest power of a_j dividing $x_P(\{a_i\})$ since $x_P(\{a_i\}) = x_Q(\{a_i\})$. Hence $x_Q(\{a_k\}) \nmid x_P(\{a_i\})$. Note that $a_j \nmid x_Q(\{a_j\})$ by statement (*) in the proof of Theorem 5.1. Therefore, $x_Q(\{a_k\}) \nmid \text{lcm}\{x_P(\{a_i\}), x_Q(\{a_j\})\}$, contrary to the formula (38).

Types 1-6 tell us that if $h(C_1) = h(D_1)$ and $C_1 \subseteq D_1$ in \mathcal{S}_Q then $C_1 = D_1$.

Similar to (I), we can prove that

(II) If $h(C_1) = h(D_1)$ and $C_1 \supseteq D_1$ then $C_1 = D_1$.

(III) If $h(C_1) = h(D_1)$ then $C_1 \subseteq D_1$ or $C_1 \supseteq D_1$.

Assume that $C_1 \parallel D_1$. Let $\{a_i\} \vee \{a_j\} = C_1$ and $\{a_k\} \vee \{a_e\} = D_1$. Then $C = C_1 \vee D_1 = \{a_i\} \vee \{a_j\} \vee \{a_k\} \vee \{a_e\} \not\supseteq \{a_i\} \vee \{a_j\} = C_1$. Thus by (I), we have that $h(C_1) < h(C)$. This follows that

$$\text{lcm}\{x_Q(\{a_i\}), x_Q(\{a_j\})\} < \text{lcm}\{x_Q(\{a_i\}), x_Q(\{a_j\}), x_Q(\{a_k\}), x_Q(\{a_e\})\}.$$

Therefore,

$$x_Q(\{a_k\}) \nmid \text{lcm}\{x_Q(\{a_j\}), x_Q(\{a_j\})\} \text{ or } x_Q(\{a_e\}) \nmid \text{lcm}\{x_Q(\{a_j\}), x_Q(\{a_j\})\}, \quad (39)$$

and formula (39) imply that

$$h(D_1) = \text{lcm}\{x_Q(\{a_k\}), x_Q(\{a_e\})\} \nmid \text{lcm}\{x_Q(\{a_i\}), x_Q(\{a_j\})\} = h(C_1),$$

i.e., $h(C_1) \neq h(D_1)$, a contradiction.

From (I), (II) and (III), we know that the map h is injective. □

The following example will illustrate Theorem 5.3.

Example 5.1 Let

$\mathcal{S}_P = \{\{a_1, a_2, a_3, a_4\}, \{a_2, a_3, a_4\}, \{a_1, a_3, a_4\}, \{a_3, a_4\}, \{a_2, a_3\}, \{a_1, a_4\}, \{a_1\}, \{a_2\}, \{a_3\}, \{a_4\}, \emptyset\}$. It is easy to see that $(\mathcal{S}_P, \subseteq)$ is a super-atomic lattice in $\mathcal{L}(4)$. Denote \mathcal{C}_P as a labeling of \mathcal{S}_P defined by (20). Then $M_{\mathcal{S}_P, \mathcal{C}_P} = \{a_2^3 a_3^4 a_4^3, a_1^3 a_3^3 a_4^4, a_1^2 a_2 a_4^2, a_1 a_2^2 a_3^2\}$. Clearly, the labeling \mathcal{C}_P is a strong coordinatization.

Let $\mathcal{S}_Q = \mathcal{S}_P - \{\{a_2, a_3, a_4\}\}$. Then $\mathcal{S}_P \succ \mathcal{S}_Q$ in lattice $\mathcal{L}(4)$. Clearly $x_Q(\{a_i\}) = \triangle_Q(\{a_i\})$ for any $i \in \{1, 2, 3, 4\}$. Then by Theorem 5.3,

$$M_{\mathcal{S}_Q, \mathcal{C}_Q} = \{a_2^2 a_3^3 a_4^2, a_1^3 a_3^3 a_4^4, a_1^2 a_2 a_4^2, a_1 a_2^2 a_3^2\} \text{ and } LCM(M_{\mathcal{S}_Q, \mathcal{C}_Q}) \cong \mathcal{S}_Q.$$

6 Conclusions

This paper studied monomial ideals by their associated lcm-lattices. It first introduced notions of weak coordinatizations which have weaker hypotheses than coordinatizations, and showed the characterizations of all such weak coordinatizations which partly answer the problem arisen by Mapes in [10]. It then defined a finite super-atomic lattice in $\mathcal{L}(n)$ which are used to investigate the structures of $\mathcal{L}(n)$ and to identify a specific labeling, given by us, of finite atomic lattice is the weak coordinatizations or the coordinatizations. It will be very interesting to study a minimal free resolution of R/M by our results in the future.

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